How to calculate values of the Inverse Cumulative Distribution Function (ICDF) for discrete random variables

by Gilberto E. Urroz, February 2006

This notes expand on the definition given in the worksheet for discrete probability distributions and in the slides for the lecture on that subject.

The Cumulative Distribution Function (CDF) of any random variable \( X \) is defined by

\[
F(x) = P(X \leq x) \tag{1}
\]

In Maple this function is calculated using \( CDF(X,x) \).

The Inverse Cumulative Distribution Function (ICDF) corresponding to \( p = F(x) = P(X \leq x) \), is defined as

\[
x = F^{-1}(p) \tag{2}
\]

To calculate this value in Maple we'll use

\[
x = F^{-1}(p) = F^{-1}[P(X \leq x)] = \text{Quantile}(X,p) - 1. \tag{2b}
\]

For any other probability statement, such as,

\[
P(X \geq x) = 0.15, \tag{3}
\]

it is necessary to cast the probability statement in terms of \( P(X \leq x) \). For example, by using the probability of a complement we can write, for this case:

\[
P(X \geq x) = 1 - P(X \leq x-1) = 1 - 0.15 = 0.85. \tag{4}
\]

This follows from the fact that if event \( A \) is defined by

\[
A = \{ X \geq x \}, \tag{5}
\]

then its complement is

\[
A' = \{ X \leq x-1 \}, \tag{6}
\]

and, by definition, \( P(A) = 1 - P(A') \).

In equation [4] let's define

\[
p = P(X \leq x-1). \tag{7}
\]

Thus, the equation becomes,

\[
1 - p = 0.85, \tag{8}
\]
or $p = 0.15$. From equation [7], then, it follows that

$$P(X \leq x-1) = 0.15.$$  \[9\]

Notice the similarity between equations [9] and [1]. Thus, the equivalent of equation [2] would be:

$$x-1 = F^{-1}(0.15),$$  \[10\]

and

$$x = 1 + F^{-1}(0.15).$$  \[11\]

From equation [2b], then we can write, for equation [11]:

$$x = x = 1 + F^{-1}(0.15) = 1 + \text{Quantile}(X,0.15) - 1 = \text{Quantile}(X,0.15).$$  \[12\]

Other probability statements would have to be manipulated so that you identify a term, $\eta = P(X \leq \xi)$, and solve for $\xi = \text{Quantile}(X,\eta) - 1$. Then, solve for $x$ from the value of $\xi$.

For example, given the statement

$$P(a \leq X \leq x) = 0.20,$$  \[13\]

where $a$ is a constant, we can write

$$P(a \leq X \leq x) = P(X \leq x) - P(X \leq a-1) = F(x) - F(a-1) = 0.20.$$  \[14\]

Then, we define $\eta = F(x)$, and re-write [14] as

$$\eta = F(x) = - F(a-1) + 0.20.$$  \[15\]

The value of $F(a-1)$ can be calculated using $CDF(X,a-1)$, so that the right-hand side of equation [15] has a specific value. Then, we can solve for $x$ from [15] as follows:

$$x = F^{-1}(\eta) = \text{Quantile}(X,\eta) - 1.$$  \[16\]

Other statements involving intervals will be expanded as follows:

1. $P(a < X \leq x) = P(X \leq x) - P(X \leq a) = F(x) - F(a)$, make $\eta = F(x)$, solve for $x$ directly, i.e., $x = F^{-1}(\eta) = \text{Quantile}(X,\eta) - 1$
2. $P(a \leq X < x) = P(X \leq x-1) - P(X \leq a-1) = F(x-1) - F(a-1)$, make $\eta = F(x-1)$, solve for $x-1$ directly, i.e., $x - 1 = F^{-1}(\eta) = \text{Quantile}(X,\eta) - 1$, thus, $x = \text{Quantile}(X,\eta)$
3. $P(a < X < x) = P(X < x) - P(X < a-1) = F(x) - F(a)$, make $\eta = F(x-1)$, solve for $x-1$ directly, i.e., $x - 1 = F^{-1}(\eta) = \text{Quantile}(X,\eta) - 1$, thus, $x = \text{Quantile}(X,\eta)$