

## Lecture 29

Relevant sections in text: §3.9

### Spin correlations and quantum weirdness: Spin 1/2 systems

Consider a pair of spin 1/2 particles created in a spin singlet state. (Experimentally speaking, this can be done in a variety of ways; see your text.) Thus the state of the system is defined by the state vector

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).$$

Let us suppose that the particles propagate – undisturbed and not interacting – until they are well-separated. Particle 1 has a component of spin measured by Observer 1 and particle 2 has a component of spin measured by Observer 2. To begin, suppose both observers measure spin along the  $z$  axis. If observer 1 sees spin up, what does observer 2 see? You probably will correctly guess: spin down. But can you prove it? Well, the reason for this result is that the state of the system is an eigenvector of  $\mathbf{S}_z$  with eigenvalue zero. So, the two particles are known with certainty to have opposite values for their  $z$ -components of spin. Alternatively, you can see from the expansion of  $|\psi\rangle$  in the product basis that the only states that occur (with equal probability) are states with opposite spins. Let us see how to prove this systematically; it's a good exercise.

We can ask the question as the following sequence of simple questions. What is the probability  $P(S_{1z} = \frac{\hbar}{2})$  that observer 1 gets spin up? That's easy:\*

$$P(S_{1z} = \frac{\hbar}{2}) = |\langle ++ | \psi \rangle|^2 + |\langle +- | \psi \rangle|^2 = \frac{1}{2}.$$

Of course, there is nothing special about particle 1 compared to particle 2; the same result applies to particle 2. What's the probability for getting particle 1 with spin up and particle 2 with spin down? We have

$$P(S_{1z} = \frac{\hbar}{2}, S_{2z} = -\frac{\hbar}{2}) = |\langle +- | \psi \rangle|^2 = \frac{1}{2}.$$

\* Well, it's "easy" if you realize that, in a state  $|\psi\rangle$ , the probability for getting an eigenvalue  $a$  of an operator  $A$  is (exercise)

$$P(a) = \sum_{i=1}^d |\langle i | \psi \rangle|^2,$$

where  $|i\rangle$  are a basis for the  $d$ -dimensional subspace of vectors with eigenvalue  $a$ :

$$A|i\rangle = a|i\rangle, \quad i = 1, 2, \dots, d.$$

You can now easily infer that – in the singlet state – when particle 1 is determined to have spin up along  $z$ , then particle 2 will have spin down along  $z$  with certainty. We say that the  $S_z$  variables for particles 1 and 2 are “completely correlated”. Another way to state this is the following. If – in the singlet state – particle 1 is determined to have spin up along  $z$ , then the state vector of the system for the purposes of all subsequent measurements is given by  $|+\ -\rangle$ . A subsequent measurement of the  $z$  component of spin for particle 2 is then going to give spin-down with certainty. The foregoing discussion will work for any choice of axis, that is, it holds for any component of the spin (the same component being measured by each observer).

We imagine repeating this experiment many times. If only one observer measures  $S_z$ , then he/she will see a random sequence of spin up or spin down results with a 50-50 probability. Quantum mechanics says that this is the best you can do – you cannot predict with certainty the  $S_z$  value for each particle because the individual particle spin is not compatible with the magnitude of the total spin of the system, which is what is known with certainty. The (singlet) state vector has been specified and there is no more information to be had about the system. How do the particles “know” that they have to have a perfect correlation in their  $S_z$  components, given that they can be separated by arbitrarily large distances? EPR would like to say: each of the particles *does* have a definite value of  $S_z$  when they are created, and these values are correlated as described above, it’s just that quantum mechanics is unable to predict the  $S_z$  values for each particles in the first place and assigns them a random probability distribution.

EPR go on to argue that if, instead, quantum mechanics is the ultimate description then a paradox arises. To see this, reason as follows. If both observers make measurements, then observers 1 and 2 each see a random distribution of spin up and spin down results, with a 50-50 probability. But when they compare their data they see that each observer 1 spin up is matched with an observer 2 spin down and vice versa. Thus, in this particular situation, one might say that particle 2 has in each experimental run a definite value of  $S_z$ . (Of course, the choice of the  $z$  axis is completely arbitrary.) Now, suppose that observer 1 chooses to measure  $S_x$  instead of  $S_z$ . When observer 1 gets spin up/down along  $x$ , then the state of the system is (exercise)  $|S_{1x} = \pm\hbar/2; S_{2x} = \mp\hbar/2\rangle$  and observer 2’s  $S_z$  measurement will now yield spin up and spin down with a 50-50 probability. To see this, the probability  $P(S_{2z} = \hbar/2)$  for getting  $S_{2z} = \hbar/2$  in either state  $|S_{1x} = \pm\hbar/2; S_{2x}, \mp\hbar/2\rangle$  can be computed by

$$\begin{aligned} P(S_{2z} = \hbar/2) &= |\langle S_{1x} = \hbar/2; S_{2z} = \hbar/2 | S_{1x} = \pm\hbar/2; S_{2x}, \mp\hbar/2 \rangle|^2 \\ &\quad + |\langle S_{1x} = -\hbar/2; S_{2z} = \hbar/2 | S_{1x} = \pm\hbar/2; S_{2x}, \mp\hbar/2 \rangle|^2 \\ &= \frac{1}{2}. \end{aligned}$$

Evidently, in this case particle 2 does not have a definite value for  $S_z$  it has a 50-50

probability distribution. On the other hand, if observer 2 chose to measure  $S_x$ , with certainty the result will be opposite to the outcomes of observer 1's measurement. In other words, when observer 1 measures  $S_x$ , particle 2 can be viewed as having a definite value for  $S_x$ , but not of  $S_z$ .

Put differently we can say: (i) A measurement of one spin component yields random (50-50) values. (ii) When observer 1 measures  $S_z$  observer 2's  $S_z$  value is perfectly correlated with observer 1's measurement. (iii) When observer 1 measures  $S_x$ , observer 2's  $S_z$  values are random – uncorrelated with observer 1's results.

From this discussion it is clear that – according to the description provided by quantum mechanics – particle 2's “reality” depends upon how observer 1 does his/her measurement. One cannot assert that the spin observables were a part of the reality of each particle independent of the measurements. This contradicts the assumption of “locality”, *i.e.*, that for suitably separated particles the experiments done upon one should not affect the other.

The preceding is a pretty run-of-the-mill quantum mechanical result, ultimately stemming from the incompatibility of observables. But it is not consistent with EPR's principle of locality, which prohibits the observable properties of particle 2 to depend upon those of particle 1 when the two can be separated by an arbitrary distance. Thus we arrive at the “EPR paradox”. EPR argue that the only way out is to suppose that all the spin components of each particle are in fact well-defined, determined with certainty by the state preparation process, and quantum mechanics is just an incomplete theory with a concomitant bit of non-locality in its predictions.

However, experiments have in fact supported the view that quantum mechanics is in fact non-local in the sense described above. In what follows we sketch the analysis that supports this claim.

## Bell's theorem

Here we briefly describe Bell's attempt to follow up on the EPR conclusion. We will derive a certain prediction (an inequality satisfied by correlation functions) based upon the EPR proposal that all spin components are in fact determined with certainty and that they exist “locally”. We shall then show that this prediction disagrees with the quantum mechanical prediction. Finally we then point out that experimental results support the quantum mechanical prediction.

Suppose that when the two particles are created all their spin components are in fact defined with certainty and that the values of the spin components of one particle are not influenced by any experiments done on the second particle. Let us consider the relation between the outcomes of measurements made by observers 1 and 2 in the case where they measure spin components about arbitrarily chosen axes  $\hat{n}_1$  and  $\hat{n}_2$ , respectively. We

report the outcomes of the measurements in units of  $\hbar/2$  and we suppose that observer 1's result is given by a function  $A(\hat{n}_1, \lambda) = \pm 1$  and observer 2's result is given by a function  $B(\hat{n}_2, \lambda) = \pm 1$ , where  $\lambda$  is any set of ("hidden") variables needed to uniquely determine the outcome.\* Of course quantum mechanics does not have access to  $\lambda$ , but if the EPR idea is correct, there should be some set of variables that determine the spin components for each particle when they are created. These variables will, in general, change from one creation to the next; after all we know that each of the spin components is observed to randomly fluctuate between the two possible values with a 50-50 probability. We characterize the changing  $\lambda$  over many experiments in a very general way: using a probability density  $\rho(\lambda)$  (whose specific form will be irrelevant for the purposes of this discussion). So, for example, the average value of  $A(\hat{n}, \lambda)$  over many experimental runs is

$$\langle A(\hat{n}) \rangle = \int d\lambda \rho(\lambda) A(\hat{n}, \lambda).$$

We next introduce the fact that when the two observers' detectors are both aligned ( $\hat{n}_1 = \hat{n}_2$ ) we get equal and opposite values for  $A$  and  $B$  – with certainty:

$$A(\hat{n}, \lambda) = -B(\hat{n}, \lambda).$$

We first focus on the average of the product of the two measurements over many experimental runs:

$$\begin{aligned} \langle A(\hat{n}_1)B(\hat{n}_2) \rangle &= \int d\lambda \rho(\lambda) A(\hat{n}_1, \lambda) B(\hat{n}_2, \lambda) \\ &= - \int d\lambda \rho(\lambda) A(\hat{n}_1, \lambda) A(\hat{n}_2, \lambda). \end{aligned}$$

Notice that this is essentially the correlation function of the two spin measurements.

Next, recall that the unusual "non-local" behavior appeared when we considered measurements of the spin of one particle in one direction only but varied the direction of spin being measured for the other particle. So, introducing yet another direction,  $\hat{n}_3$ , we have (exercise)

$$\langle A(\hat{n}_1)B(\hat{n}_2) \rangle - \langle A(\hat{n}_1)B(\hat{n}_3) \rangle = - \int d\lambda \rho(\lambda) [1 - A(\hat{n}_2, \lambda)A(\hat{n}_3, \lambda)] A(\hat{n}_1, \lambda) A(\hat{n}_2, \lambda),$$

where we used the fact that (because we measure in units of  $\hbar/2$ )

$$[A(\hat{n}, \lambda)]^2 = 1.$$

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\* The locality assumption is used when we say that  $A$  doesn't depend upon  $\hat{n}_2$  and that  $B$  doesn't depend upon  $\hat{n}_1$ .

It is now easy to establish the estimate (exercise):

$$|\langle A(\hat{n}_1)B(\hat{n}_2) \rangle - \langle A(\hat{n}_1)B(\hat{n}_3) \rangle| \leq \int d\lambda \rho(\lambda) [1 - A(\hat{n}_2, \lambda)A(\hat{n}_3, \lambda)].$$

You can prove this estimate by observing that for any  $f(\lambda)$

$$|\int d\lambda f(\lambda)| \leq \int d\lambda |f(\lambda)|.$$

Next, using

$$\int d\lambda \rho(\lambda) = 1,$$

we get

$$|\langle A(\hat{n}_1)B(\hat{n}_2) \rangle - \langle A(\hat{n}_1)B(\hat{n}_3) \rangle| \leq 1 + \langle A(\hat{n}_2)B(\hat{n}_3) \rangle.$$

This is a version of the *Bell inequality*. It holds for a very wide range of “local, hidden variable” theories in which all observables are compatible. We now show that quantum mechanics, of course, violates this inequality so that one can, in effect, experimentally determine whether quantum mechanics is complete.