

## Lecture 27

Relevant sections in text: §3.6, 3.7

### Angular momentum addition in general

We can generalize our previous discussion of 2 spin 1/2 systems as follows. Suppose we are given two angular momenta  $\vec{J}_1$  and  $\vec{J}_2$  (e.g., two spins, or a spin and an orbital angular momentum, or a pair of orbital angular momenta). We can discuss both angular momenta at once using the direct product space as before, with a product basis  $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ . We represent the operators on product vectors as

$$\vec{J}_1(|\alpha\rangle \otimes |\beta\rangle) = (\vec{J}|\alpha\rangle) \otimes |\beta\rangle,$$

and

$$\vec{J}_2(|\alpha\rangle \otimes |\beta\rangle) = |\alpha\rangle \otimes (\vec{J}|\beta\rangle),$$

and extend to general vectors by linearity. The product basis  $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$  is the basis corresponding to the commuting observables provided by  $(J_1^2, J_2^2, J_{1z}, J_{2z})$ .

The total angular momentum is defined by

$$\vec{J} = \vec{J}_1 + \vec{J}_2.$$

A set of commuting observables that includes the total angular momentum is provided by the operators  $(J_1^2, J_2^2, J^2, J_z)$ . Note that both bases are eigenvectors of  $J_1^2$  and  $J_2^2$  since these commute with all components of the individual and total angular momentum (exercise). We also note that product eigenvectors  $|j_1, j_2, m_1, m_2\rangle$  are in fact eigenvectors of  $J_z$  with eigenvalues given by  $m = m_1 + m_2$  since

$$J_z|j_1, j_2, m_1, m_2\rangle = (J_{1z} + J_{2z})|j_1, j_2, m_1, m_2\rangle = (m_1 + m_2)\hbar|j_1, j_2, m_1, m_2\rangle.$$

But we will have to take linear combinations of product basis vectors to get total angular momentum vectors – eigenvectors of  $J^2$ .

The basis of total angular momentum eigenvectors are denoted  $|j_1, j_2, j, m\rangle$ . For given values of  $j_1$  and  $j_2$ , it can be shown that (see the text)

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \dots, j_1 + j_2 - 1, j_1 + j_2$$

with (as usual)

$$m = -j, -j + 1, \dots, j - 1, j.$$

The two sets of commuting observables defining each kind of basis are not all compatible. In particular,  $J_{1z}$  and  $J_{2z}$  do not commute with  $J^2$ . So the set of total angular momentum

eigenvectors will be distinct from the eigenvectors of the individual angular momenta. Total angular momentum eigenvectors,  $|j_1, j_2, j, m\rangle$  can be expressed as linear combinations of  $|j_1, j_2, m_1, m_2\rangle$  where the superposition will go over various  $m_1, m_2$  values. Of course one can also expand  $|j_1, j_2, m_1, m_2\rangle$  in terms of  $|j_1, j_2, j, m\rangle$  where the superposition will be over various  $j, m$  values. The coefficients in these superpositions are known as the *Clebsch-Gordan coefficients*. We have worked out a very simple example of all this in the case of a pair of spin 1/2 systems. There is a general theory of Clebsch-Gordan coefficients which we shall not have time to explore. Instead we will briefly visit another, relatively simple, and relatively important example.

### Example: A particle in 3-d with spin 1/2

We now apply these ideas and obtain some formulas describing the angular momentum of a particle moving in 3-d with spin 1/2. Of course this is a model of a non-relativistic electron so it is extremely important.

To begin, we specify the Hilbert space of states. Using the tensor product construction, we can view it as the space of formal linear combinations of the product basis:

$$|\vec{x}, \pm\rangle = |\vec{x}\rangle \otimes |S_z, \pm\rangle.$$

A general state is of the form

$$|\psi\rangle = \int d^3x (a_+(\vec{x})|\vec{x}, +\rangle + a_-(\vec{x})|\vec{x}, -\rangle).$$

As usual,  $|a_{\pm}(\vec{x})|^2$  is the probability density for finding the particle at  $\vec{x}$  with spin up/down along the  $z$  axis. Note that the normalization condition is

$$\int d^3x (|a_+|^2 + |a_-|^2) = 1.$$

Alternatively, we can characterize the state in terms of its components in the basis defined above. In this case we organize the information into a 2-component column vector whose entries are complex valued functions. This gadget is known as a *spinor field*:

$$\begin{pmatrix} \langle \vec{x}, + | \psi \rangle \\ \langle \vec{x}, - | \psi \rangle \end{pmatrix} = \begin{pmatrix} a_+(\vec{x}) \\ a_-(\vec{x}) \end{pmatrix}.$$

The position, momentum and spin operators are defined on the product basis as follows:

$$\vec{X}(|\vec{x}\rangle \otimes |S_z, \pm\rangle) = \vec{x}|\vec{x}, \pm\rangle, \quad \vec{P}(|\vec{x}\rangle \otimes |S_z, \pm\rangle) = (\vec{P}|\vec{x}\rangle) \otimes |S_z, \pm\rangle,$$

$$\vec{S}(|\vec{x}\rangle \otimes |S_z, \pm\rangle) = |\vec{x}\rangle \otimes (\vec{S}|S_z, \pm\rangle).$$

This implies that, on the spinor fields the position and momentum operators do their usual thing ( $\vec{X}$  multiplies,  $\vec{P}$  differentiates) on each component function while the spin operators do their usual thing via  $2 \times 2$  matrices. For example, the orbital angular momentum is  $\vec{L} = \vec{X} \times \vec{P}$  and acts as

$$\vec{X} \times \vec{P} \begin{pmatrix} a_+(\vec{x}) \\ a_-(\vec{x}) \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{i} \vec{x} \times \nabla a_+(\vec{x}) \\ \frac{\hbar}{i} \vec{x} \times \nabla a_-(\vec{x}) \end{pmatrix},$$

and

$$S_x \begin{pmatrix} a_+(\vec{x}) \\ a_-(\vec{x}) \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a_-(\vec{x}) \\ a_+(\vec{x}) \end{pmatrix}.$$

We can now define the total angular momentum of the system as the operator

$$\vec{J} = \vec{L} + \vec{S}.$$

As usual, because  $\vec{L}$  and  $\vec{S}$  commute and satisfy the angular momentum commutation relations, we have

$$[J_a, J_b] = i\hbar \epsilon_{abc} J_c,$$

so the total angular momentum has all the general properties we deduced previously. For example,  $\vec{J}$  generates rotations of the system as a whole, while  $\vec{L}$  only generates rotations of position and momentum, and while  $\vec{S}$  only generates rotations of the spin. We can only simultaneously diagonalize  $L^2$ ,  $S^2$ ,  $J^2$  and one component, say,  $J_z$ . Setting  $\vec{J}_1 = \vec{L}$  and  $\vec{J}_2 = \vec{S}$  we have  $j_1 = l = 0, 1, 2, \dots$  and  $j_2 = s = \frac{1}{2}$ . For a state specified by  $|l, s = 1/2, j, m\rangle$  we then have that the possible values for  $j$  are

$$j = l - \frac{1}{2}, l + \frac{1}{2}.$$

How are the total angular momentum eigenvectors related to the original product eigenvectors (eigenvectors of  $L_z$  and  $S_z$ )? We will sketch the construction. Begin with the eigenvectors of total angular momentum with the maximum value of angular momentum:

$$|j_1 = l, j_2 = \frac{1}{2}, j = l + \frac{1}{2}, m = l + \frac{1}{2}\rangle.$$

Given  $j_1$  and  $j_2$ , there is only one (linearly independent) product eigenvector which has the appropriate eigenvalue for  $J_z$ , namely

$$|j_1 = l, j_2 = \frac{1}{2}, m_1 = l, m_2 = \frac{1}{2}\rangle,$$

so we have

$$|j_1 = l, j_2 = \frac{1}{2}, j = l + \frac{1}{2}, m = l + \frac{1}{2}\rangle = |j_1 = l, j_2 = \frac{1}{2}, m_1 = l, m_2 = \frac{1}{2}\rangle.$$

To get the lower eigenvalues of  $J_z$  we can apply the ladder operator

$$J_- = L_- + S_-$$

to this ket and normalize it. The details can be found in the text. Similarly, we can also start with the minimum values for  $j = l - \frac{1}{2}$  and  $m = -l + \frac{1}{2}$  and construct the states with the other  $m$  values using the raising operator. In this way we can express all the total angular momentum eigenvectors in terms of the product eigenvectors. The result is not too complicated. We have, for  $j = l \pm \frac{1}{2}$ ,

$$|j_1 = l, j_2 = \frac{1}{2}, j = l \pm \frac{1}{2}, m\rangle = \frac{1}{\sqrt{2l+1}} \left[ \pm \sqrt{l \pm m + \frac{1}{2}} |l, m - \frac{1}{2}\rangle \otimes |+\rangle + \sqrt{l \mp m + \frac{1}{2}} |l, m + \frac{1}{2}\rangle \otimes |-\rangle \right].$$

In terms of spinor fields we get the *total angular momentum spinors*  $\mathbf{Y}_{j,m}(\theta, \phi)$ , representing the total angular momentum eigenvectors by taking components in the product basis:

$$\mathbf{Y}_{j,m} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} \pm \sqrt{l \pm m + \frac{1}{2}} Y_{m_l=m-\frac{1}{2}}^l(\theta, \phi) \\ \sqrt{l \mp m + \frac{1}{2}} Y_{m_l=m+\frac{1}{2}}^l(\theta, \phi) \end{pmatrix}.$$

The upper (lower) component of this column vector determines the probability density for finding a particle at various angular locations and with a spin up (down) along  $z$ , given that its total angular momentum is known with certainty to have the values specified by  $j = l \pm \frac{1}{2}$  and  $m$ .