Lecture 26

Relevant sections in text: §3.6, 3.7

Two spin 1/2 systems: observables

We have constructed the 4-d Hilbert space of states for a system consisting of two spin 1/2 particles. We built the space from the basis of "product states" corresponding to knowing the spin along z for each particle with certainty. General states were, however, not necessarily products but rather superpositions of such. How are the observables to be represented as Hermitian operators on this space? To begin, let us consider the spin observables for each of the particles. Call them $\vec{S}_i = (\vec{S}_1, \vec{S}_2)$. We define them on product states via

$$\vec{S}_1(|\alpha\rangle \otimes |\beta\rangle) = (\vec{S}|\alpha\rangle) \otimes |\beta\rangle,$$

and

$$\vec{S}_2(|\alpha\rangle \otimes |\beta\rangle) = |\alpha\rangle \otimes (\vec{S}|\beta\rangle).$$

Here the operators $\vec{S}$ are the usual spin 1/2 operators (acting on a two-dimensional Hilbert space) that we have already discussed in some detail.

If $|\alpha\rangle$ is an eigenvector of spin along some axis, then so is $|\alpha\rangle \otimes |\beta\rangle$ for any $|\beta\rangle$. This means that if we know the spin component along the chosen axis with certainty for particle one then we get an eigenvector of the corresponding component of $\vec{S}_1$, as we should. The same remarks apply to particle 2. The action of $\vec{S}_1$ and $\vec{S}_2$ are defined on general vectors by expanding those vectors in a product basis, such as we considered above, and then using linearity to evaluate the operator term by term on each vector in the expansion. Sometimes one writes

$$\vec{S}_1 = \vec{S} \otimes I, \quad \vec{S}_2 = I \otimes \vec{S}$$

to summarize the above definition.

The two spin operators $\vec{S}_1$ and $\vec{S}_2$ commute (exercise) and have the same eigenvalues as their 1-particle counterparts (exercise). In this way we recover the usual properties of each particle, now viewed as subsystems.

Total angular momentum

There are other observables that can be defined for the two particle system as a whole. Consider the *total* angular momentum $\vec{\mathbf{S}}$, defined by

$$\vec{\mathbf{S}} = \vec{S}_1 + \vec{S}_2.$$

You can easily check that this operator is Hermitian and that

$$[\mathbf{S}_k, \mathbf{S}_l] = i\hbar\epsilon_{klm}\mathbf{S}_m,$$

so it does represent the angular momentum. Indeed, this operator generates rotations of the two particle system as a whole. The individual spin operators $\vec{S}_1$ and $\vec{S}_2$ only generate rotations of their respective subsystems.

Using our general theory of angular momentum we know that we can find a basis of common eigenvectors of $\mathbf{S}^2$ and any one component, say, $\mathbf{S}_z$. Let us write these as $|s, m_s\rangle$, where

$$\mathbf{S}^2|s, m_s\rangle = s(s+1)\hbar^2|s, m_s\rangle, \quad \mathbf{S}_z|s, m_s\rangle = m\hbar|s, m_s\rangle.$$

Let us define

$$|\pm, \pm\rangle = |S_z, \pm\rangle \otimes |S_z, \pm\rangle.$$

This product basis physically corresponds to states in which the z component of spin for each particle is known with certainty. In the following we will find the total angular momentum eigenvalues and express the eigenvectors in the product basis $|\pm, \pm\rangle$.

To begin with, it is clear that eigenvectors of $\mathbf{S}_z$ are in fact the basis of product vectors since (with $m_1 = \pm\frac{1}{2}$, $m_2 = \pm\frac{1}{2}$)

$$\mathbf{S}_z|m_1, m_2\rangle = (S_{1z} + S_{2z})|m_1, m_2\rangle = (m_1 + m_2)\hbar|m_1, m_2\rangle.$$

We see that $m = -1, 0, 1$ with $m = 0$ being doubly degenerate (exercise). From our general results on angular momentum it is clear that the only possible values for the total spin quantum number are $s = 0, 1$. From this we can infer that the $m = \pm 1$ eigenvectors must be $\mathbf{S}^2$ eigenvectors with $s = 1$, but we may need linear combinations of the $m = 0$ product eigenvectors to get $\mathbf{S}^2$ eigenvectors. To see why the vectors $|++\rangle$ and $|--\rangle$ must be also $\mathbf{S}^2$ eigenvectors one reasons as follows. Our general theory guarantees us the existence of a basis of simultaneous $\mathbf{S}^2$ and $\mathbf{S}_z$ eigenvectors. It is easy to see that the $|++\rangle$ and $|--\rangle$ are the only eigenvectors (up to normalization) with $m = \pm 1$, since any other vectors can be expanded in the product basis and this immediately rules out any other linear combinations (exercise). Therefore, these two vectors must be the $\mathbf{S}^2$ eigenvectors. Because they have $m = \pm 1$ and we know that $s = 0, 1$ it follows that the $|++\rangle$ and $|--\rangle$ vectors are $\mathbf{S}^2$ eigenvectors with $s = 1$.

To determine the linear combinations of the $m = 0$ product vectors $|+-\rangle$ and $|-+\rangle$ that yield $\mathbf{S}^2$ eigenvectors we use the angular momentum ladder operators:

$$\mathbf{S}_\pm = S_x \pm iS_y = S_{1\pm} + S_{2\pm}.$$

If we apply $\mathbf{S}_-$ to the eigenvector

$$|s = 1, m = 1\rangle = |++\rangle,$$

we get (exercise)

$$|s = 1, m = 0\rangle = \mathbf{S}_-|s = 1, m = 1\rangle = \mathbf{S}_-|++\rangle = (S_{1-} + S_{2-})|++\rangle = \frac{1}{\sqrt{2}}(|-+\rangle + |+-\rangle).$$

The other eigenket $|0,0\rangle$ must be orthogonal to this vector as well as to the other eigenkets, $|++\rangle$ and $|--\rangle$, from which its formula follows (exercise). All together, we find the *total* angular momentum eigenvectors, $|s,m\rangle$, are related to the *individual* angular momentum (product) eigenkets by:

$$
\begin{aligned}
|1,1\rangle &= |++\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \\
|1,-1\rangle &= |--\rangle, \\
|0,0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).
\end{aligned}
$$

These vectors form an orthonormal basis for the Hilbert space, so they are all the linearly independent eigenvectors of $\mathbf{S}^2$ and $\mathbf{S}_z$. The eigenstates with $s=1$ are called the *triplet* states and the eigenstate with $s=0$ is the *singlet state*. Notice that by combining two systems with half-angular momentum we end up with a system that allows integer angular momentum only.

A lengthier – but more straightforward – derivation of the eigenvectors $|s,m_s\rangle$ arises by simply writing the 4×4 matrix for $\mathbf{S}^2$ in the basis of product vectors $|\pm\pm\rangle$, and solving its eigenvalue problem. This is a good exercise. To get this matrix, you use the formula

$$\mathbf{S}^2 = \mathbf{S}_z^2 + +\hbar\mathbf{S}_z + \mathbf{S}_+\mathbf{S}_-.$$

It is straightforward to deduce the matrix elements of this expression among the product states since each of the operators has a simple action on those vectors.

Notice that the states of definite *total* angular momentum, $|s,m_s\rangle$, are not all the same as the states of definite individual angular momentum, say, $|\pm\pm\rangle$. This is because the total angular momentum is not compatible with the individual angular momentum. For example,

$$[\mathbf{S}^2, S_{1i}] = [S_1^2 + S_2^2 + 2(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}), S_{1i}] \neq 0.$$

A couple of complete sets of commuting observables are given by $(S_1^2, S_2^2, S_{1z}, S_{2z}))$ and $(S_1^2, S_2^2, \mathbf{S}^2, \mathbf{S}_z)$. The eigenvectors of the first set are the product basis $|m_1, m_2\rangle = |\pm\pm\rangle$, representing states in which each individual spin angular momentum state is known with certainty. The eigenvectors of the second set are given by $|s,m_s\rangle$, representing states in which the total angular momentum is known with certainty.

A remark on identical particles

Let us remark that there is yet another postulate in quantum mechanics that deals with *identical particles*. These are particles that are intrincsically alike (same mass, spin, electric charge, *etc.*). Thus, for example, all electrons are identical, though of course they can be in different states. This does not mean that electrons cannot be distinguished *literally*, since we can clearly distinguish between an electron here on earth and one on the sun. These are two electrons in different (position) states. But we view these particles as interchangeable in the sense that if one took the electron from the sun and replaced it with the one here on Earth (putting them in the respective states) when you weren't looking, then you couldn't tell. This intrinsic indistinguishability of identical particles opens up the possibility of having the states of multi-particle systems reflect this symmetry under particle interchange. This symmetry is modeled as a discrete, unitary transformation which exchanges particles. The postulate of quantum mechanics (which can more or less be *derived* from relativistic quantum field theory) is that particles with integer spin ("bosons") should be invariant under this unitary transformation ("even" under exchange) and they should change sign ("odd under exchange") if they particles have half-integer spin ("fermions").

You can see that the *total* spin states of two spin 1/2 systems are in fact even and odd under particle interchange. If the two particles are identical and no other degrees of freedom are present, then one must use the anti-symmetric singlet state only. Of course, real particles have translational degrees of freedom and the state will reflect that. Using position wave functions to characterize these degrees of freedom, one again can consider the symmetric and anti-symmetric combinations. Only the *total* state vector must have the appropriate symmetry. For example, consider the ground state for two electrons in a Helium atom. The position space ground wave function is symmetric under particle interchange. Thus the ground state must be a singlet. Excited states can, however by described by symmetric spin states if the "position part" of the state vector is anti-symmetric under particle interchange.