Lecture 25

Relevant sections in text: §3.6, 3.7

Position representation of angular momentum operators

We have seen that the position operators act on position wave functions by multiplication and the momentum operators act by differentiation. We can combine these two results and, using spherical polar coordinates (r, θ, ϕ), get a useful position wave function representation for the angular momentum operators. We have

$$L_x\psi(r,\theta,\phi) = \frac{\hbar}{i}\left(-\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi\right)\psi(r,\theta,\phi),$$

$$L_y\psi(r,\theta,\phi) = \frac{\hbar}{i}\left(-\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi\right)\psi(r,\theta,\phi)$$

$$L_z\psi(r,\theta,\phi) = \frac{\hbar}{i}\partial_\phi\psi(r,\theta,\phi).$$

You can see that L_z is particularly simple – it clearly generates "translations" in ϕ, which are rotations about the z axis, of course. The other two components of $\vec{L}$ also generate rotations about their respective axes. They do not take such a simple form because spherical polar coordinates give the z axis special treatment.

Combining these results we have, in addition,

$$L^2\psi(r,\theta,\phi) = -\hbar^2\left(\frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\right)\psi(r,\theta,\phi).$$

You may recognize that this last result is, up to a factor of $-\hbar^2 r^2$, the angular part of the Laplacian. This result arises from the identity (see text)

$$L^2 = r^2P^2 - (\vec{X}\cdot\vec{P})^2 + i\hbar\vec{X}\cdot\vec{P},$$

where $r^2 = X^2 + Y^2 + Z^2$, so that (exercise)

$$P^2\psi(r,\theta,\phi) = -\hbar^2\nabla^2\psi(r,\theta,\phi) = -\hbar^2(\frac{1}{\hbar^2r^2}L^2 + \partial_r^2 + \frac{2}{r}\partial_r)\psi(r,\theta,\phi).$$

Thus we get, in operator form, the familiar decomposition of kinetic energy into a radial part and an angular part.

Orbital angular momentum eigenvalues and eigenfunctions; spherical harmonics

A good way to see what is the physical content of the orbital angular momentum eigenvectors is to study the position probability distributions in these states. Thus we consider the position wave functions

$$\psi_{lm} = \langle\vec{x}|l,m\rangle$$

corresponding to orbital angular momentum eigenvectors. These are simultaneous eigenfunctions of L^2 and L_z, so they satisfy

$$L_z\psi_{lm_l} = m_l\hbar\psi_{lm_l}, \quad L^2\psi_{lm_l} = l(l+1)\hbar^2\psi_{lm_l},$$

where – on general grounds –

$$l = 0, \frac{1}{2}, 1, \dots,$$

and

$$m_l = -l, -l+1, \dots l-1, l.$$

We note that the angular momentum eigenfunctions will always involve an arbitrary multiplicative function of the radius r. This is because the angular momentum differential operators only take derivatives in the angular directions. What this means physically is that the states of definite angular momentum will always have a degeneracy. This should not surprise you: just specifying the angular momentum of a state of a particle is not expected to completely determine the particle's state.*

We now argue that the half-integer possibility does not occur for orbital angular momentum. First, we note that it is quite easy to find L_z eigenfunctions in spherical polar coordinates since $L_z = -i\hbar\partial_\phi$. Evidently,

$$\psi_{lm_l} = f_{lm_l}(r,\theta)e^{im_l\phi}.$$

Immediately we see that m_l can only be an integer – otherwise ψ_{lm} will not be a continuous function. Now, discontinuous wave functions are not inherently evil. Indeed, there are plenty of discontinuous functions in the Hilbert space. But these functions will fail to be differentiable and hence will not in the domain of the momentum and angular momentum operators. This is a contradiction since, of course, we are trying to construct the angular momentum eigenfunctions which, by definition, are in the domain of the operators. Thus we can conclude immediately that, for orbital angular momentum, we can only have (at most)

$$l = 0, 1, 2, \dots, \quad \text{and} \quad m_l = -l, -l+1, \dots l-1, l.$$

As we shall see, all of the indicated values do in fact arise. (By the way, in your homework assignment you will find a problem which gives an alternative argument for this result.)

Orbital angular momentum eigenvalues and eigenfunctions; spherical harmonics

Having solved the L_z equation we now must solve the L^2 equation, which is an ordinary

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* For example, consider the classical motion of a particle with vanishing angular momentum. The motion is such that position and momentum vectors are parallel, but are otherwise arbitrary.

differential equation for $f_{lm_l}(r,\theta)$:

$$\frac{1}{\hbar^2}L^2 f_{lm_l} = \left(\frac{m_l^2}{\sin^2\theta} - \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)\right) f_{l,m_l} = l(l+1)f_{lm_l}.$$

The solutions of this equation are the associated Legendre polynomials $P_{l,m_l}(\cos\theta)$ and the angular momentum eigenfunctions are thus of the form

$$\psi_{l,m_l}(r,\theta,\phi) = \tilde{f}_{l,m_l}(r)Y_{l,m_l}(\theta,\phi),$$

where the $Y_{l,m}(\theta,\phi)$ are the *spherical harmonics* and the functions $\tilde{f}_{lm_l}(r)$ are the "integration constants" for the solution to the purely angular differential equations. See your text for detailed formulas for the spherical harmonics. Note that all non-negative integer values are allowed for l. As discussed earlier, the functions $\tilde{f}_{lm_l}(r)$ are not determined by the angular momentum eigenvalue problem. Typically these functions are fixed by requiring the wave function to be also an eigenfunction of another observable which commutes with L^2 and L_z, *e.g.,* the energy in a central force problem. In any case, we will assume that

$$\int_0^\infty dr\, r^2 |f_{l,m_l}(r)|^2 = 1.$$

This way, with the conventional normalization of the spherical harmonics:

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi\, \sin^2\theta Y^*_{l',m_l'}(\theta,\phi)Y_{l,m_l}(\theta,\phi) = \delta_{ll'}\delta_{m_l m_l'},$$

we have that

$$\langle l', m_l'|l, m_l\rangle = \int_0^\infty dr\, r^2 \int_0^\pi d\theta \int_0^{2\pi} d\phi, \sin^2\theta \psi^*_{l',m_l'}(r,\theta,\phi)\psi_{l,m_l}(r,\theta,\phi) = \delta_{ll'}\delta_{m_l m_l'}.$$

For a state of definite angular momentum $|l, m_l\rangle$ we see that the angular dependence of the probability distribution is completely determined by the spherical harmonics. The radial dependence of the probability distribution is not determined by the value of angular momentum unless other requirements are made upon the state.

Addition of angular momentum: Two spin 1/2 systems

We now will have a look at a rather important and intricate part of angular momentum theory involving the combination of two (or more) angular momenta. We will primarily focus on the problem of making a quantum model for a system consisting of two distinguishable spin 1/2 particles (ignoring all but their spin degrees of freedom). Again, the idea is simply to combine two copies of our existing model of a spin 1/2 system. The technology we shall need has already been introduced in our discussion of the direct product construction. We shall take this opportunity to review the construction in the context of the problem of combining – or "adding" – two spin 1/2 angular momenta.

For a system of two spin 1/2 particles, *e.g.,* an electron and a positron, we can imagine measuring the component of spin for each particle along a given axis, say the z axis. Obviously there are 4 possible outcomes (exercise). Having made these measurements, we can denote the states in which these spin values are known with certainty by

$$|S_z,+\rangle\otimes|S_z,+\rangle,\,|S_z,+\rangle\otimes|S_z,-\rangle,\,|S_z,-\rangle\otimes|S_z,+\rangle,\,|S_z,-\rangle\otimes|S_z,-\rangle.$$

Here the first factor of the pair always refers to "particle 1" and the second factor refers to "particle 2". We view these vectors as an orthonormal basis for the direct product Hilbert space of states of 2 spin 1/2 particles. We thus consider the 4-d Hilbert space of formal linear combinations of these 4 basis vectors. An arbitrary vector $|\psi\rangle$ is given by

$$|\psi\rangle = a_{++}|S_z,+\rangle\otimes|S_z,+\rangle+a_{+-}|S_z,+\rangle\otimes|S_z,-\rangle+a_{-+}|S_z,-\rangle\otimes|S_z,+\rangle+a_{--}|S_z,-\rangle\otimes|S_z,-\rangle.$$

Here the scalar multiplication is assigned to the pair as a whole, but by definition it can be assigned to either of the factors in the pair as well. If you wish you can view the scalars $a_{\pm\pm}$ as forming a column vector with 4 rows; the squares of these scalars give the various probabilities for the outcome of the S_z measurement for each particle. Other bases are possible, corresponding to other experimental arrangements, *e.g.,* S_x for particle 1 and S_y for particle 2.