Lecture 24

Relevant sections in text: §3.5, 3.6

Angular momentum eigenvalues and eigenvectors (cont.)

Next we show that the eigenvalues of J^2 are non-negative and bound the magnitude of the eigenvalues of J_z. One way to see this arises by studying the relation

$$J^2 - J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+) = \frac{1}{2}(J_-^\dagger J_- + J_+^\dagger J_+).$$

Now, for any operator A and vector $|\psi\rangle$ we have that (exercise)

$$\langle\psi|A^\dagger A|\psi\rangle \geq 0,$$

so that for any vector $|\psi\rangle$ (in the domain of the squared angular momentum operators) (exercise)

$$\langle\psi|J^2 - J_z^2|\psi\rangle \geq 0.$$

Assuming the eigenvectors $|a, b\rangle$ are not of the "generalized" type, *i.e.,* are normalizable, we have

$$0 \leq \langle a, b|J^2 - J_z^2|a, b\rangle = a - b^2,$$

and hence

$$a \geq 0, \quad -\sqrt{a} \leq b \leq \sqrt{a}.$$

The ladder operators increase/decrease the b value of the eigenvector with out changing a. Thus by repeated application of these operators we can violate the inequality above unless there is a maximum and minimum value for b such that application of J_+ and J_-, respectively, will result in the zero vector. Moreover, if we start with an eigenvector with a minimum (maximum) value for b, then by successively applying J_+ (J_-) we must hit the maximum (minimum) value. As shown in your text, these requirements lead to the following results. The eigenvalues a can only be of the form

$$a = j(j+1)\hbar^2,$$

where $j \geq 0$ can be a non-negative integer or a half integer only:

$$j = 0, 1/2, 1, 3/2, \ldots.$$

For an eigenvector with a given value of j, the eigenvalues b are given by

$$b = m\hbar,$$

where

$$m = -j, -j+1, \ldots, j-1, j.$$

Note that if j is an integer then so is m, and if j is a half-integer, then so is m. Note also that for a fixed value of j there are $2j+1$ possible values for m. The usual notational convention is to denote angular momentum eigenvectors by $|j, m\rangle$, with j and m obeying the restrictions described above.

The preceding arguments show how the self-adjointness and commutation relations of angular momentum give plenty of information about their spectrum. We note that these are necessary conditions, *e.g.,* the magnitude of angular momentum must be determined via an integer or half-integer, but this does not mean that all these possibilities will occur. As we shall see, for orbital angular momentum only the integer possibility is utilized. For the spin 1/2 system, a single value $j = 1/2$ is utilized. We will discuss this in a little more detail next.

Spin systems in general

Let us note that the spin 1/2 observables, being angular momentum operators, must have eigenvectors/eigenvalues obeying the general results we have just derived. Indeed, you can easily see that with $j = 1/2$ we reproduce the standard results on the spectrum of the spin operators. For example we have

$$J^2 \leftrightarrow S^2 = \frac{3}{4}\hbar^2 I,$$

which has eigenvalues

$$\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2.$$

Given that $j = \frac{1}{2}$ we have

$$m = -\frac{1}{2}, \frac{1}{2},$$

so that the eigenvalues for $J_z \leftrightarrow S_z$ are $\pm\hbar$, as they should be.

The "1/2" in "spin 1/2" comes from the fact that $j = 1/2$ for all states of interest in this physical system. We can generalize this to other values of j. We speak of a particle or system having spin s if it admits angular momentum operators which act on a Hilbert space of states all of which have the same eigenvalue for S^2, that is, all of which have the same value for $j = s$. For a system with spin-s and no other degrees of freedom the Hilbert space of states has dimension $2s+1$ (exercise) and the operator representing the squared-magnitude of the spin is given by (exercise)

$$S^2 = s(s+1)\hbar^2 I.$$

Orbital angular momentum

In nature it appears that angular momentum comes in two types when we use a "particle" description of matter. First there is the intrinsic "spin" angular momentum carried by an elementary particle. The spin of a particle is fixed once and for all (although the spin state is not) and is part of the essential attribute that makes a particle what it is. Second, there is the "orbital" angular momentum which arises due to the motion of the particle in space. Both of these types of angular momentum are to some extent unified when using the presumably more fundamental description of matter and its interactions afforded by quantum field theory.

The orbital angular momentum of a particle is represented by the operator

$$\vec{L} = \vec{X} \times \vec{P},$$

where $\vec{X}$ is the position operator relative to some fixed origin. Let us note that the ordering of the operators $\vec{X}$ and $\vec{P}$ is unambiguous since the cross product only brings commuting operators into play. For example,

$$L_z = XP_y - YP_x.$$

The formula above for $\vec{L}$ is familiar from classical mechanics, but it can be justified using the angular momentum commutation relations. For example, you can check by a straightforward computation that

$$[L_x, L_y] = [YP_z - ZP_y, ZP_x - XP_z] = i\hbar(XP_y - YP_x) = i\hbar L_z.$$

The other angular momentum commutation relations follow in a similar fashion.

Orbital angular momentum and rotations

To further justify this form of the orbital angular momentum, we can study its role as infinitesimal generator of rotations. Let us consider an infinitesimal rotation about the z-axis. The putative generator is

$$L_z = XP_y - YP_x.$$

We can study the action of L_z on states by computing its action on the positions basis,

$$|\vec{x}\rangle = |x, y, z\rangle, \quad \vec{R}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle.$$

An infinitesimal rotation by an angle ϵ is given by

$$D(\epsilon) = I - \frac{i}{\hbar}\epsilon L_z + \mathcal{O}(\epsilon^2),$$

so that, to first order in ϵ,

$$\begin{aligned} D(\epsilon)|x, y, z\rangle &= [I - \frac{i}{\hbar}\epsilon(xP_y - yP_x)]|x, y, z\rangle + \mathcal{O}(\epsilon^2) \\ &= |x - \epsilon y, y + \epsilon x, z\rangle + \mathcal{O}(\epsilon^2). \end{aligned}$$

Here we used the fact that momentum generates translations. Now recall the following geometric fact: under an infinitesimal rotation about an axis $\hat{n}$ by an angle ϵ the position vector (indeed, any vector) transforms as

$$\vec{x} \to \vec{x} + \epsilon\hat{n} \times \vec{x} + \mathcal{O}(\epsilon^2).$$

Choosing $\hat{n}$ along the z axis, we can compare this formula with the change of the position eigenvector under the infinitesimal transformation generated by L_z. We see that the position eigenvector's eigenvalue rotates properly (at least infinitesimally).

It is not hard to see that under a finite (*i.e.,* non-infinitesimal) rotation about the z-axis we have

$$\begin{aligned} e^{-\frac{i}{\hbar}\theta L_z}|x, y, z\rangle &= |x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta, z\rangle \\ &= |R(\hat{k}, \theta\rangle \cdot \vec{x}\rangle. \end{aligned}$$

You can prove this by iterating the infinitesimal transformation, for example. Since the z axis is arbitrary, we have in fact proved (exercise)

$$e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle = |R(\hat{n}, \theta\rangle \cdot \vec{x}\rangle.$$

This implies that (exercise)

$$e^{\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}\vec{X}e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}} = R(\hat{n}, \theta) \cdot \vec{X},$$

which can be checked by evaluating it on the position basis. Therefore we have that (exercise)

$$X(e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle) = R(\hat{n}, \theta) \cdot \vec{x}(e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle),$$

so that the rotation operator on the Hilbert space maps eigenvectors of position to eigenvectors with the rotated position:

$$e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{x}\rangle = |R(\hat{n}, \theta) \cdot \vec{x}\rangle.$$

From this result we have the position wave functions rotating properly (exercise):

$$e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}\psi(\vec{x}) = \langle\vec{x}|e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\psi\rangle = \psi(R^{-1}(\hat{n}, \theta) \cdot \vec{x}).$$

An identical set of results can be obtained for the momentum operators and their eigenvectors and momentum wave functions. This is a satisfactory set of results since the momentum *vector* should behave in the same way as the position vector under rotations. We have, in particular

$$\begin{aligned} D(\hat{n}, \theta)|\vec{p}\rangle &= e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}|\vec{p}\rangle = |R(\hat{n}, \theta)\vec{p}\rangle, \\ e^{\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}}\vec{P}e^{-\frac{i}{\hbar}\theta\hat{n}\cdot\vec{L}} &= R(\hat{n}, \theta)\vec{P}. \end{aligned}$$