Lecture 23

Relevant sections in text: §3.2, 3.5

Spin precession as a rotation

It is enlightening to return to the dynamical process of spin precession in light of our new results on rotations. You will recall that a spin system with magnetic moment $\vec{\mu}$ when placed in a uniform magnetic field $\vec{B}$ can be described by the Hamiltonian

$$H = -\vec{\mu} \cdot \vec{B}, \quad \text{where} \quad \vec{\mu} = \mu\vec{S}.$$

You will recall that the behavior of the spin observables could be viewed as precession about $\vec{B}$, *i.e.,* a continuously developing (in time) rotation about an axis along $\vec{B}$. We can now see this result immediately. Let $\hat{n}$ be a unit vector along $\vec{B}$, so that

$$H = -\mu B\hat{n} \cdot \mathbf{S}.$$

This means that the time evolution operator is

$$U(t, t_0) = e^{\frac{i}{\hbar}(t-t_0)\mu B\hat{n}\cdot\vec{S}}.$$

This operator represents a rotation about $\hat{n}$ by an "angle" $\mu B(t - t_0)$, which is exactly our previous result for the dynamics.

Note that, while the physical observables are precessing with frequency μB, the state vector itself has is precessing at half the frequency since, *e.g.,* it takes a 4π rotation to get the state vector to return to its initial value. It is possible to experimentally "see" this difference in frequencies (thereby confirming the projective representation being used) by a pair of spin 1/2 systems, one of which propagates freely and one of which travels through a region with a magnetic field. The latter spin will precess according to the time it spends in the magnetic field. The two particles can be brought together to form an interference pattern. The interference pattern depends upon the relative phase of the two particles. If the magnetic field region is set up just right, you can arrange for the second particle to change its state vector by a minus sign by the time it leaves the magnetic field region. The interference pattern that you see confirms this fact. See your text for details.

Angular momentum in general

We can deduce quite a lot about the angular momentum $\vec{J}$ of any system just knowing that it is represented by 3 self-adjoint operators satisfying the angular momentum commutation relations. To begin with, it is clear that the 3 components of $\vec{J}$ are not compatible so that, generally speaking, one will not be able to determine more than one component with certainty. Indeed, the only state in which 2 or more components of $\vec{J}$ are known with certainty is an eigenvector of all components with eigenvalue zero, *i.e.,* a state with vanishing angular momentum. To see this, suppose that $|\alpha\rangle$ is an eigenvector of J_x and J_y, then it is easy to see from

$$[J_x, J_y] = i\hbar J_z,$$

that $|\alpha\rangle$ is an eigenvector of J_z with eigenvalue 0. You can easily see that this same argument can now be used to show that $|\alpha\rangle$ has zero eigenvalue for all three components. Thus, if there is any angular momentum in the system at all, at most one component can be known with certainty in any state. When we consider states with a definite value for a component of $\vec{J}$, we usually call that component J_z, by convention. But it is important to realize that there is nothing special about the z-direction; one can find eigenvectors for any one component of $\vec{J}$ (*cf.* spin 1/2).

We next observe that the (squared) magnitude of the angular momentum,

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

is a Hermitian operator that is compatible with any component J_i. To see this is a very simple computation:

$$[J^2, J_k] = \sum_l \left(J_l[J_l, J_k] + [J_l, J_k]J_l\right) = i\hbar \sum_{l,m} \epsilon_{lkm}(J_l J_m + J_m J_l) = 0,$$

where the last equality follows from the antisymmetry of ϵ_{lkm}.* We will assume that J^2 is self-adjoint. Consequently, there exists an orthonormal basis of simultaneous eigenvectors of J^2 and any one component of $\vec{J}$ (usually denoted J_z). Physically, this means that while the 3 components of angular momentum are not compatible, there exists a complete set of states in which the magnitude of angular momentum and one component of angular momentum are known with certainty.

Angular momentum eigenvalues and eigenvectors

Of course, given an observable represented as an operator, the most pressing business is to understand the spectral properties of the operator since its spectrum determines the possible outcomes of a measurement of the observable and the (generalized) eigenvectors are used to compute the probability distribution of the observable in a given state. In our case we have defined angular momentum as operators satisfying

$$\vec{J} = \vec{J}^\dagger, \quad [J_l, J_m] = i\hbar\epsilon_{lmn} J_n.$$

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* The quantity in parenthesis is symmetric under $l \leftrightarrow m$ while ϵ_{lkm} is anti-symmetric when this interchange is performed. This guarantees that each term in the double sum will be canceled by another term in the double sum.

Just from these relations alone there is a lot we can learn about the spectral properties of angular momentum.

We assume that each of the operators J_i and J^2 admit eigenvectors. Let us study the angular momentum eigenvalues and eigenvectors, the latter being simultaneous eigenvectors of J_z and J^2. We write

$$J^2|a,b\rangle = a|a,b\rangle, \quad J_z|a,b\rangle = b|a,b\rangle,$$

The possible values of a and b can be deduced much in the same way as the spectrum of the Hamiltonian for an oscillator can be deduced using the raising and lowering operators. To this end we define the angular momentum *ladder operators*

$$J_\pm = J_x \pm iJ_y, \quad J_\pm^\dagger = J_\mp.$$

Of course, these two operators contain the same physical information as J_x and J_y. In terms of the ladder operators, the angular momentum commutation relations can be expressed as (exercise)

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad [J_\pm, J^2] = 0, \quad [J_\pm, J_\mp] = \pm 2\hbar J_z.$$

From these relations we can see that the vector $J_\pm|a,b\rangle$ satisfies (exercise)

$$J^2(J_\pm|a,b\rangle) = a(J_\pm|a,b\rangle), \quad J_z(J_\pm|a,b\rangle) = (b\pm\hbar)(J_\pm|a,b\rangle).$$

Thus, when acting on angular momentum eigenvectors (eigenvectors of J^2 and J_z), the ladder operators preserve the magnitude of the angular momentum but increase/decrease the z component by a “quantum of angular momentum” $\hbar$.

To be continued…