

Lecture 18

Relevant sections in text: §2.3

Stationary states and classical mechanics

Here we use the oscillator to illustrate a very key point about the relation between classical and quantum mechanics.

The stationary states we just studied do not provide any explicitly dynamical behavior. This is not a specific feature of the oscillator, but a general feature of stationary states in quantum mechanics. This is, at first sight, a little weird compared to classical mechanics. Think about it: the probability distributions for position and momentum are time independent in any state of definite energy. In classical mechanics the position and momentum (and the energy) can, at each moment of time, be determined with certainty; the values for the position and momentum oscillate in time in every state but the ground state.* In light of the incompatibility of the position, momentum and energy observables in the quantum description, one cannot directly compare the classical and quantum predictions in the excited states. The quantum predictions are purely statistical, involving repeated state preparations – states specified only by their energy – and measurements of various observables. If we want to compare the quantum and classical descriptions we need to ask the right questions of classical mechanics — this means statistical questions. Let us pause for a moment to expand on this.

In classical mechanics every solution of the equations of motion is a state of definite energy (assuming energy is conserved). Fix an energy E . Let us ask the same question that we ask in quantum theory: Given the energy what is the probability for finding the particle at various locations? To answer this question we define the probability for finding the classical particle to be in the interval $[x, x + dx]$ to be proportional to the time spent in that region. (The proportionality constant is used to normalize the probability distribution.) For a classical oscillator at the point x , a displacement dx takes place in the time dt where (exercise)

$$dt = \frac{dx}{\sqrt{\frac{2E}{m} - \omega^2 x^2}}.$$

Here it is understood that x lies between the two classical turning points, where

$$E = \frac{1}{2}m\omega^2 x^2.$$

* In the ground state, the classical motion is of course trivial — the position (relative to the equilibrium position) and the momentum vanish. In the quantum ground state the position and momentum have Gaussian probability distributions centered at the vanishing values. For macroscopic values of m and ω the widths of these distributions are negligible.

Up to a constant normalization factor, this defines the probability density $P(x)$ for the classical oscillator:

$$P(x) = (\text{const.}) \frac{1}{\sqrt{\frac{2E}{m} - \omega^2 x^2}}.$$

(The probability density vanishes outside the turning points.) For $E \neq 0$, the resulting probability density is strongly peaked near the classical turning points of the motion and relatively small and flat near the equilibrium position. This just reflects the fact that, because the oscillator moves slowly near the turning points and rapidly at the equilibrium position, it is more likely to find the particle near the turning points. In general, a quantum oscillator does not have anything like this classical probability distribution. First of all, it oscillates in a non-trivial way because of the Hermite polynomials. Furthermore, while the probability density is exponentially damped outside of the turning points, the probability distribution is not identically zero there. However, it can be shown that the quantum probability distribution *does* approximate the classical distribution in the limit of “large energy”. Here, “large” means sufficiently “macroscopic”, *i.e.*, $E_n \gg \hbar\omega$ so that $n \gg 1$. To see this one simply computes $|u_n|^2$ for a large n . The result is a very rapidly oscillating wavefunction; the oscillations occur about an average curve which approaches the classical probability distribution as n gets larger and larger. For any finite interval of x , a large enough n will result in a probability for finding the particle in that interval to agree with the classical prediction to any desired accuracy. To see this quickly, simply ask your favorite computer math software to plot the graphs!

We have seen that the probability distribution for position predicted by quantum mechanics approaches that predicted by classical *statistical mechanics* in the limit of “large quantum numbers”. This result is satisfying, but is far from the complete story of the relation of the classical and quantum features of the oscillator. A much better way to model the classical oscillator does not use stationary states but instead minimum uncertainty “coherent states” (see the exercises at the end of chapter 2). Recall that the excited states are not minimum uncertainty states in position and momentum. The coherent states are minimum uncertainty states; for macroscopic mass and frequency the oscillator has a small enough uncertainty in energy, position and momentum and a suitable time dependence to model a classical oscillator. In particular, because the coherent states aren’t stationary (except for the ground state), the position and momentum probability distributions can mimic their classical counterparts – sinusoidal oscillations. We will not explore the coherent states here. We do note that the ground state of the oscillator is one such state. For macroscopic values of m and ω the width of the ground state Gaussian position probability distribution – controlled by $x_0 = \frac{\hbar}{m\omega}$ is truly negligible compared to macroscopic length scales, so you can see at least in this state that classical behavior is recovered.

Oscillator Dynamics

To explicitly see the oscillatory behavior of an oscillator in quantum mechanics one needs to use non-stationary states, that is, superpositions of energy eigenstates, as the initial state. Let us examine this feature using the Heisenberg picture. The principal job is to get the basic observables at time t in terms of their representation at some fixed initial time, say, $t_0 = 0$. We have

$$X(0) \equiv X, \quad P(0) \equiv P.$$

We need to compute $X(t)$ and $P(t)$. We can do this directly using

$$X(t) = e^{\frac{i}{\hbar}Ht} X(0) e^{-\frac{i}{\hbar}Ht}, \quad P(t) = e^{\frac{i}{\hbar}Ht} P(0) e^{-\frac{i}{\hbar}Ht},$$

and

$$H = H(t) = \frac{P^2(t)}{2m} + \frac{1}{2}m\omega^2 X^2(t) = H(0) = \frac{P^2(0)}{2m} + \frac{1}{2}m\omega^2 X^2(0).$$

To evaluate $X(t)$ and $P(t)$ you can either expand the exponentials in power series and, by studying the general term, try to deduce a closed form expression for the result. There are other tricks as well for manipulating the similarity transformation. However, a somewhat easier way to get expressions for $X(t)$ and $P(t)$ is to directly solve the Heisenberg equations. Using (exercise)

$$[X(t), P(t)] = i\hbar I,$$

we have

$$\begin{aligned} i\hbar \frac{d}{dt} X(t) &= [X(t), H] = i\hbar \frac{P(t)}{m}, \\ i\hbar \frac{d}{dt} P(t) &= [P(t), H] = -i\hbar m\omega^2 X(t), \end{aligned}$$

So that the Heisenberg equations can be written as

$$\frac{d}{dt} X(t) = [X(t), H] = \frac{P(t)}{m}, \quad \frac{d}{dt} P(t) = [P(t), H] = -m\omega^2 X(t).$$

These are mathematically the same as the classical Hamilton equations of motion, and are as easily solved. Taking account the initial conditions at $t = 0$, we have

$$\begin{aligned} X(t) &= (\cos \omega t) X(0) + \left(\frac{1}{m\omega} \sin \omega t\right) P(0), \\ P(t) &= (\cos \omega t) P(0) - (m\omega \sin \omega t) X(0). \end{aligned}$$

In the Heisenberg picture, the time dependence in physical predictions (*i.e.*, probability distributions) is obtained using these observables and a fixed state vector. As an example, let us compute the statistical mean of the position and momentum at time t . Taking diagonal matrix elements of the Heisenberg operators, we get

$$\langle X \rangle(t) = (\cos \omega t) \langle X \rangle(0) + \left(\frac{1}{m\omega} \sin \omega t\right) \langle P \rangle(0),$$

$$\langle P \rangle(t) = (\cos \omega t) \langle P \rangle(0) - (m\omega \sin \omega t) \langle X \rangle(0).$$

From the above equations you can see that we get the usual sort of behavior expected of a harmonic oscillator with mass m and frequency ω . Indeed, we see that the expectation values behave exactly as do the classical position and momentum, in accord with Ehrenfest's theorem.

Of course, if the system is in a stationary state $|n\rangle$ then expectation values do not change in time. This seems paradoxical in light of our result shown above. The resolution of this paradox in the above example is that, for stationary states, the initial expectation values vanish, *e.g.*,

$$\langle X \rangle(t) = \langle X \rangle(0) = 0, \quad \langle P \rangle(t) = \langle P \rangle(0) = 0.$$

As an exercise see if you can see how the expectation values of, say, X^2 manages to be time independent in a stationary state.