

*Lecture 15**Relevant sections in text: §2.2***Compatible/Incompatible observables in the Heisenberg picture**

Let us also note that the notion of compatible/incompatible observables is left undisturbed by the transition to the Heisenberg picture. This is because the commutator transforms as (exercise)

$$[A(t), B(t)] = [U^\dagger A U, U^\dagger B U] = U^\dagger [A, B] U.$$

(Note any two operators related by a unitary/similarity transformation will have this commutator property.) Thus, if \mathbf{A} and \mathbf{B} are (in)compatible in the Schrödinger picture they will be (in)compatible (at each time) in the Heisenberg picture. Note also that the commutator of two observables in the Schrödinger picture — which is i times another observable — makes the transition to the Heisenberg picture just as any other Schrödinger observable, namely, via the unitary transformation

$$A \rightarrow U^\dagger A U.$$

Unitary transformations in general

Let us note that while our discussion was phrased in the context of time evolution, the same logic can be applied to any unitary transformation. For example, for a particle moving in one dimension one can view the effect of translations as either redefining the state vector, leaving the operator-observables unchanged:

$$|\psi\rangle \rightarrow T_a |\psi\rangle, \quad A \rightarrow A$$

or equivalently as redefining the observables, with the state vectors unchanged:

$$A \rightarrow T_a^\dagger A T_a, \quad |\psi\rangle \rightarrow |\psi\rangle.$$

Note, in particular, that the position and momentum operators change in the expected way under a translation (exercise – you played with this stuff in the homework):

$$T_a^\dagger X T_a = x + a, \quad T_a^\dagger P T_a = p.$$

Heisenberg equations

We saw that the conventional Schrödinger equation is really just a consequence of the relation between the time evolution operator and its infinitesimal generator in the context of the Schrödinger picture:

$$i\hbar \frac{d}{dt} U(t, t_0) = H(t) U(t, t_0) \quad \iff \quad i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle.$$

Given a Hamiltonian, this equation is the starting point for investigating quantum dynamics in the Schrödinger picture. We can now ask: what is the analog of this in the Heisenberg picture? In the Heisenberg picture, dynamical evolution occurs through the operator-observables, which for simplicity we assume to be time independent in the Schrödinger picture. We have

$$A(t) = U^\dagger(t, t_0)AU(t, t_0).$$

Differentiating both sides and using our basic differential equation for $U(t, t_0)$ we get

$$\begin{aligned} \frac{d}{dt}A(t) &= -\frac{1}{i\hbar}U^\dagger(t, t_0)H(t)AU(t, t_0) + U^\dagger(t, t_0)A\frac{1}{i\hbar}H(t)U(t, t_0) \\ &= -\frac{1}{i\hbar}U^\dagger(t, t_0)H(t)U(t, t_0)U^\dagger(t, t_0)AU(t, t_0) + \frac{1}{i\hbar}U^\dagger(t, t_0)AU(t, t_0)U^\dagger(t, t_0)H(t)U(t, t_0) \\ &= \frac{1}{i\hbar}[A(t), H_{Heis}(t)]. \end{aligned}$$

Here we have introduced the Heisenberg picture version of the Hamiltonian:

$$H_{Heis}(t) = U^\dagger(t, t_0)H(t)U(t, t_0).$$

If the (Schrödinger) Hamiltonian is time independent, as is often the case, then we have (exercise)

$$H_{Heis}(t) = H(t) = H, \quad \frac{\partial H}{\partial t} = 0.$$

This is pretty important to keep in mind.

The equation

$$\frac{d}{dt}A(t) = \frac{1}{i\hbar}[A(t), H_{Heis}(t)]$$

is the *Heisenberg equation of motion* for the Heisenberg operator $A(t)$. Given a Hamiltonian it is a differential equation that, in principle, can be solved to find the Heisenberg operator corresponding to an observable at time t , given initial conditions

$$A(t_0) = A.$$

Given $A(t)$ one gets the time dependence of probability distributions in the usual way. The outcome of a measurement of the observable represented (at time t) by $A(t)$ is one of the eigenvalues a_i . The probability for getting a_i (assuming no degeneracy) at time t is

$$P(a_i, t) = |\langle a_i, t | \psi, t_0 \rangle|^2 = |\langle a_i, t_0 | U(t, t_0) | \psi, t_0 \rangle|^2,$$

where $|a_i, t\rangle$ is the eigenvector of $A(t)$ with eigenvalue a_i and $|\psi, t_0\rangle$ is the state vector for the system.

The Heisenberg equation can make certain results from the Schrödinger picture quite transparent. For example, just by taking expectation values on both sides of the equation using the (single, time-independent) state vector it is apparent that (exercise)

$$\frac{d}{dt}\langle \mathbf{A} \rangle(t) = \frac{1}{i\hbar}\langle [A, H] \rangle.$$

Note that here the notation $[A, H]$ means the observable corresponding to the commutator, which is well defined in either picture. Similarly, it is easy to see that operators that commute with the Hamiltonian (assumed time-independent for simplicity, so that $H_{Heis} = H$) at one time t_0 are *constants of the motion*:

$$\frac{d}{dt}A(t) = \frac{1}{i\hbar}[A(t), H] = U^\dagger[A(t_0), H]U = 0.$$

Thus, conserved quantities satisfy

$$A(t) = A(t_0) = A.$$

You can also see this result directly. If an observable commutes with H at one time, $t = t_0$ say, then it will not change in time since

$$A(t) = U^\dagger(t, t_0)A(t_0)U(t, t_0) = U^\dagger(t, t_0)U(t, t_0)A(t_0) = A(t_0).$$

Evidently, a conserved quantity has a time independent probability distribution since the operator in the Heisenberg picture does not change in time.

Functions of observables

We have seen that the time evolution operator defines, via a similarity transformation, the Heisenberg operators at time t :

$$A(t) = U^\dagger A U.$$

If we have an observable that is a function of A , $F(A)$ say, we have, of course

$$F(A)(t) = U^\dagger F(A) U.$$

It is important to note that one can also express this as

$$F(A)(t) = F(A(t)) \equiv F(U^\dagger A U).$$

To see this, we use the spectral decomposition:

$$F(A) = \sum_a F(a) |a\rangle \langle a|,$$

so that

$$U^\dagger F(A)U = \sum_a F(a)U^\dagger|a\rangle\langle a|U = \sum_a F(a)|a(t)\rangle\langle a(t)| = F(A(t)).$$

Heisenberg Picture Dynamics of a Particle in a potential

For a particle with position (at one time, say $t = 0$) \mathbf{X} and momentum \mathbf{P} (these are spatial vectors made of operators – we are going back to our original notation), we consider a Hamiltonian of the form

$$H = \frac{P^2}{2m} + V(\mathbf{X}).$$

Note that this operator does not depend upon time, so it is both the Schrödinger and Heisenberg Hamiltonian. In particular, we have that

$$H(t) = H(0) = H = \frac{P(t)^2}{2m} + V(\mathbf{X}(t)).$$

This result can be viewed as a mathematical version of the conservation of energy in the Heisenberg picture.

Let us compute the Heisenberg equations for $\mathbf{X}(t)$ and momentum $\mathbf{P}(t)$. Evidently, to do this we will need the commutators of the position and momentum with the Hamiltonian. To begin, let us consider the canonical commutation relations (CCR) at a fixed time in the Heisenberg picture. Using the general identity

$$[A(t), B(t)] = U^\dagger(t, t_0)[A(t_0), B(t_0)]U(t, t_0),$$

we get (exercise)

$$[X^i(t), X^j(t)] = 0 = [P_i(t), P_j(t)], \quad [X^i(t), P_j(t)] = i\hbar\delta_j^i I.$$

In other words, the Heisenberg position and momentum operators obey the CCR at any fixed time. Note that this means the uncertainty relation between position and momentum is time independent — a fact you can also prove in the Schrödinger picture (exercise).

It is now straightforward to compute

$$[X^i(t), H] = \frac{i\hbar}{m}P^i(t),$$

so that

$$\frac{dX^i(t)}{dt} = \frac{1}{m}P^i(t).$$

Thus one relates the momentum and velocity of the particle; a result that is a bit more tricky to establish in the Schrödinger picture. To compute the Heisenberg equations for the momentum we need to compute (exercise)

$$[P_i(t), H] = [P_i(t), V(\mathbf{X}(t))].$$

Probably the simplest way to do this is to use

$$[P_i(t), V(\mathbf{X}(t))] = U^\dagger(t, 0)[P_i, V(\mathbf{X})]U(t, 0) = -i\hbar U^\dagger(t, 0) \frac{\partial V}{\partial x^i}(\mathbf{X})U(t, 0) = -i\hbar \frac{\partial V}{\partial x^i}(\mathbf{X}(t)).$$

Here we have used

$$[P_i, V(\mathbf{X})] = -i\hbar \frac{\partial V}{\partial x^i}(\mathbf{X}),$$

which can be verified by checking it on the position eigenvector basis using the definition of P_i as the generator of infinitesimal translations (good exercise). All together we get

$$\frac{dP_i(t)}{dt} = -\frac{\partial V}{\partial x^i}(\mathbf{X}(t)).$$

Using the Heisenberg equations for $\mathbf{X}(t)$, we can write the Heisenberg equation for $\mathbf{P}(t)$ as

$$\frac{d^2 X^i(t)}{dt^2} = F^i(\mathbf{X}(t)),$$

where

$$F^i(\mathbf{X}(t)) = -\frac{\partial V}{\partial x^i}(\mathbf{X}(t)),$$

can be viewed as the quantum representation of the force at time t in the Heisenberg picture. This is a quantum version of Newton's second law.

From this result it is tempting to believe that a quantum particle is actually behaving just like a classical particle. After all, the basic observables obey the same equations of motion in the two theories. Of course this is not true, if only because it is not possible to know both position and momentum with statistical certainty in the quantum theory. In the next section we will take a closer look at this issue.

As a elementary example, let us consider a free particle in 1-d, $V = 0$. The Heisenberg equations are

$$\frac{dX(t)}{dt} = \frac{P(t)}{m}, \quad \frac{dP(t)}{dt} = 0,$$

with solution (setting $t_0 = 0$)

$$X(t) = X(0) + \frac{P(0)}{m}t, \quad P(t) = P(0).$$

Here $X(0) = X$ and $P(0) = P$ are the operators we discussed previously in the Schrödinger picture. The momentum is evidently a constant of the motion: the momentum probability distribution for a free particle is time independent. The position probability distribution changes in time, *e.g.*,

$$\langle X \rangle(t) = \langle X \rangle(0) + \frac{1}{m} \langle P \rangle(0).$$