

Lecture 13

Relevant sections in text: §2.1

Example: Spin 1/2 in a uniform magnetic field.

Let us consider the dynamical evolution of an electronic spin in a (uniform) magnetic field. We ignore the translational degrees of freedom of the electron. The electron magnetic moment is an observable which we represent as

$$\boldsymbol{\mu} = - \left(\frac{e}{mc} \right) \mathbf{S},$$

(Here $e > 0$ is the magnitude of the electron electric charge.) The Hamiltonian for a magnetic moment in a (uniform, static) magnetic field \mathbf{B} is taken to be

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = \left(\frac{e}{mc} \right) \mathbf{S} \cdot \mathbf{B}.$$

Let us choose the z axis along \mathbf{B} so that

$$H = \left(\frac{eB}{mc} \right) S_z.$$

Evidently, S_z eigenvectors are energy eigenvectors. Thus the S_z eigenvectors are stationary states.

Let us consider the time evolution of a general state,

$$|\psi(t_0)\rangle = a|+\rangle + b|-\rangle, \quad |a|^2 + |b|^2 = 1.$$

We have

$$\begin{aligned} |\psi(t)\rangle &= e^{-\frac{i}{\hbar} \frac{eB}{mc} S_z} |\psi(t_0)\rangle \\ &= a e^{-i\omega t/2} |+\rangle + b e^{i\omega t/2} |-\rangle, \end{aligned}$$

where

$$\omega = \frac{eB}{mc}.$$

From this formula you can easily see that if the initial state is an S_z eigenvector, then it remains so for all time. To see dynamical evolution, we pick an initial state that is not an energy eigenvector. For example, suppose that at $t = 0$, $a = b = \frac{1}{\sqrt{2}}$, *i.e.*, $|\psi(0)\rangle = |S_x, +\rangle$.

What is the probability for getting $S_x = \pm \frac{\hbar}{2}$ at time t ? We have (exercise)

$$Prob(S_x = \frac{\hbar}{2}) = |\langle S_x, + | \psi(t) \rangle|^2 = \cos^2\left(\frac{\omega t}{2}\right),$$

$$Prob(S_x = -\frac{\hbar}{2}) = |\langle S_x, - | \psi(t) \rangle|^2 = \sin^2\left(\frac{\omega t}{2}\right).$$

Thus an effect of the magnetic field is to cause the x -component of the spin to periodically “flip”. One can visualize the behavior of the spin by following its expectation value in time. Still using $|\psi(0)\rangle = |S_x, +\rangle$ we have (good exercise)

$$\langle S_x \rangle(t) = \langle \psi(t) | S_x | \psi(t) \rangle = \frac{\hbar}{2} \cos \omega t, \quad \langle S_y \rangle(t) = \langle \psi(t) | S_y | \psi(t) \rangle = \frac{\hbar}{2} \sin \omega t, \quad \langle S_z \rangle(t) = 0.$$

Thus, on the average, the spin vector precesses about the axis determined by the magnetic field, and in the plane orthogonal to it, with angular velocity $\omega = \frac{eB}{mc}$.

Time rate of change of expectation values

Let us consider how expectation values change in time. We have (assuming that the observable of interest does not depend upon time, so that its operator representative does not change in time)

$$\frac{d}{dt} \langle A \rangle(t) = \frac{d}{dt} \langle \psi(t) | A | \psi(t) \rangle.$$

Using

$$\frac{d}{dt} |\psi(t)\rangle = \frac{1}{i\hbar} H |\psi(t)\rangle, \quad \frac{d}{dt} \langle \psi(t) | = -\frac{1}{i\hbar} \langle \psi(t) | H,$$

we get (exercise)

$$\frac{d}{dt} \langle A \rangle(t) = \frac{1}{i\hbar} \langle [A, H] \rangle(t).$$

Thus, at least on the average, dynamical evolution of an observable is controlled by its incompatibility with H . Of course, if A and H are compatible, then they admit a common basis of eigenvectors whence the probability distribution for A is time independent for the same reason it is for H (see below). As an exercise you can use the result shown above to derive the time rate of change of the expectation values that we studied in the spin 1/2 precession example in the last lecture.

Conservation laws

We have already seen (assuming $\frac{\partial H}{\partial t} = 0$) that H (usually meaning the energy) is conserved in the sense that the probability distribution for H is time-independent. In a special class of states, the stationary states, the energy is known with certainty for all time and *all* other probability distributions are time independent. However, it is possible to have conserved quantities besides the energy of course without choosing a stationary state. We say that an observable A is conserved if its probability distribution is always time independent – irrespective of the initial state of the system. It is not hard to see that A is conserved if and only if it is compatible with H :

$$[A, H] = 0.$$

The “if” part of the statement is proved as follows. If the commutator vanishes, then the basis of energy eigenvectors can be chosen to also be A eigenvectors. Let us denote those eigenvectors by $|k\rangle$, where

$$A|k\rangle = a_k|k\rangle, \quad H|k\rangle = E_k|k\rangle.$$

As before, the initial state can be expanded as

$$|\psi, t_0\rangle = \sum_k c_k |k\rangle,$$

where $|c_k|^2$ is the probability for getting a_k and/or E_k upon measurement of the observable (represented by) A and/or H . Because the basis is built from H eigenvectors we still have

$$|\psi, t\rangle = \sum_k c_k e^{-\frac{i}{\hbar} E_k (t-t_0)} |k\rangle,$$

where

$$|c_k e^{-\frac{i}{\hbar} E_k (t-t_0)}|^2 = |c_k|^2$$

is the probability for getting a_k at time t . Thus the probability distribution for A is time-independent.

The “only if” part of the statement is proved as follows. Note that if the probability distribution for A is to be time-independent then, using the results of the previous section, we must have

$$\langle \psi, t_0 | [A, H] | \psi, t_0 \rangle = 0$$

for all possible initial state vectors $|\psi, t_0\rangle$. As we have mentioned before, if an operator on a complex Hilbert space has vanishing diagonal matrix elements then it is the zero operator.

Time-Energy Uncertainty Principle

We have seen that (for a time-independent Hamiltonian) energy eigenvectors define stationary states, so that if the energy is known with certainty at one time, then the state is unchanged for all time, *i.e.*, all probability distributions are time independent. Physical attributes of the system will evolve in time only if the initial state of the system is a superposition of energy eigenvectors with different energies. Of course, in such an initial state the energy is not known “with certainty”, but rather has a non-trivial probability distribution. Thus a the statistical uncertainty in energy is related to the time rate of change of (probability distributions of) observables. The infamous *time-energy uncertainty principle* relates the time scale Δt for a significant change of a given (initial) state to the

statistical uncertainty of the energy ΔE in that state (other misleading slogans notwithstanding). You will see that this uncertainty principle is a little different than, *e.g.*, the position-momentum uncertainty principle, though it does take the form

$$\Delta t \Delta E \sim \hbar.$$

To be continued . . .