

*Lecture 11**Relevant sections in text: §1.7, 2.1***Product observables**

We have seen how to build the Hilbert space for a composite system via the tensor product construction. Let us now see how to build the observables. Let  $A_1$  be an observable for system 1 and  $B_2$  an observable for system 2. We define the corresponding observables for the composite system as follows. Consider the following linear operators on product vectors:

$$A_1 \otimes I|\alpha\rangle \otimes |\beta\rangle = (A_1|\alpha\rangle) \otimes |\beta\rangle, \quad I \otimes B_2|\alpha\rangle \otimes |\beta\rangle = |\alpha\rangle \otimes (B_2|\beta\rangle).$$

We extend the definitions of these operators to general states by expanding the state in a product basis and demanding linearity of the operator. Thus defined, these operators are Hermitian. Moreover, they commute (good exercises!):

$$[A_1 \otimes I, I \otimes B_2] = 0,$$

which means that one can still determine with statistical certainty the subsystem observables within the composite system. For example, if we are looking at 2 particles, it is possible to ascertain both particle 1's position and particle 2's momentum with arbitrarily good statistical accuracy. Likewise, when using this construction to build a model for a particle moving in 3-d from our 1-d model, we end up with all the position variables commuting among themselves and all the momentum variables commuting among themselves. Of course, the commutation relations obtained in the subsystems appear in the combined system.

*Exercise*

Show that if  $[A_1, B_1] = C_1$  then

$$[A_1 \otimes I, B_1 \otimes I] = C_1 \otimes I,$$

thus explaining the generalization of the canonical commutation relations we obtained for a particle in 3-d.

Please take note that it is conventional in many references to simply drop the  $\otimes I$  or  $I \otimes$  factors when denoting the extension of the subsystem operators to the composite system. Indeed, it is common to also drop the  $\otimes$  symbol entirely!

Also note that not every observable of the system is of the above form. As usual, any self-adjoint operator can represent an observable. Thus there will be observables that refer

to the composite system as a whole (*e.g.*, the total energy) and are not observables of either of the subsystems alone.

Returning to our example of a particle moving in 2-d. We have position operators for each of the two particles:  $X \otimes I$  and  $I \otimes Y$ . For example, we have

$$X \otimes I|x, y\rangle = X \otimes I|x\rangle \otimes |y\rangle = x|x\rangle \otimes |y\rangle.$$

Let us see how these operators act on position wave functions:

$$Y\psi(x, y) = \langle x, y|Y|\psi\rangle = y\langle x, y|\psi\rangle = y\psi(x, y),$$

where we used

$$Y|x, y\rangle = y|x, y\rangle \implies \langle x, y|Y = y\langle x, y|.$$

Similarly, we can define the momentum. For example you can check as a nice exercise that

$$P_1\psi(x, y) = (\langle x| \otimes \langle y|)P_x \otimes I|\psi\rangle = \frac{\partial}{\partial x}\psi(x, y).$$

(Hint:  $P_x$  acts on  $\langle x|$  via infinitesimal translations.)

As an example of an observable that can only be defined on the whole system, consider the total energy function for two particles moving in 1-d (equivalently, one particle moving in 2-d).

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + V(X, Y).$$

It acts on position wave functions as (exercise)

$$H\psi(x, y) = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y) + V(x, y)\psi(x, y).$$

## Dynamics: the Schrödinger picture

We now have enough tools to formulate quantum dynamics. Dynamics are characterized by the final postulate: *Time evolution is a continuous unitary transformation.* We have, of course, just studied an example of a continuous unitary transformation: Translation by an amount  $a$ . In that application  $a$  is not identified with time, though. The idea of dynamical evolution is that measurable aspects of the system are changing in time.\* In

\* For us, time will be modeled in its Newtonian form as a non-dynamical, *a priori* way of ordering events, valid in any inertial reference frame. In particular, in this framework time is not treated as an observable whose value depends upon the state of the quantum system, but instead as a special kind of parameter. Of course, there are many interesting issues here, but we will not be able to explore them further in this course. We simply assume that some suitable standard of temporal reference has been chosen once and for all.

quantum mechanics the characterization of the system at any given time can be viewed as the totality of probability distributions for all observables at that time. Time evolution, therefore, ought to correspond to a time varying change in the probability distributions. As we have noted already, all probability distributions can be computed by taking expectation values of suitable observables (*e.g.*, characteristic functions, *etc.*). Therefore, time evolution can be defined in terms of the evolution of expectation values. The rules of the game say that given an observable  $A$  the expectation value is to be computed via

$$\langle A \rangle = \langle \psi | A | \psi \rangle,$$

where  $|\psi\rangle$  is a unit vector in Hilbert space representing the state of the system and  $A$  is the operator representing the observable. We now consider doing this as time varies. At each instant of time we need to have a system of vectors and operators that can be used to make physical predictions via, say, expectation values. Our strategy will be to assume we have a single Hilbert space for all time, but we allow the mathematical identification of states and observables to vary in time. To describe mathematically how expectation values change in time we then have a number of options. Two such options are often convenient. First, we can let the operator representing the observable change in time, with the vector  $|\psi\rangle$  fixed for all time. This point of view of dynamics is called the *Heisenberg picture*, and will be discussed later. Another way of viewing dynamics, known as the *Schrödinger picture*, is based upon letting the vector  $|\psi\rangle$  change in time while holding the operator representatives of the observables fixed in time. There are infinitely many different “pictures” intermediate to these two. We shall look first at the postulate on dynamics from the point of view of the Schrödinger picture.

### The Schrödinger picture: The time evolution operator

Much as we did with spatial translations, we assume that the state vector at time  $t$ , denoted by  $|\psi, t\rangle$ , is related by a continuous, unitary transformation from the state at any earlier time  $t_0$ . Therefore we write

$$|\psi, t\rangle = U(t, t_0)|\psi, t_0\rangle.$$

Here  $|\psi, t_0\rangle$  can in principle be any unit vector.  $U$  is unitary so that the state vector remains normalized as time progresses:

$$\langle \psi, t | \psi, t \rangle = \langle \psi, t_0 | U^\dagger U | \psi, t_0 \rangle = \langle \psi, t_0 | \psi, t_0 \rangle = 1.$$

(It is not hard to show that a (bounded) operator on a complex Hilbert space vanishes if and only if all its diagonal matrix elements vanish (exercise). Since  $|\psi, t_0\rangle$  is arbitrary we see that the state vector remains normalized if and only if  $U^\dagger = U^{-1}$ , that is,  $U$  is unitary.) Given,

$$U^\dagger(t, t_0) = U^{-1}(t, t_0),$$

we naturally associate the inverse transformation as time evolution from  $t$  back to  $t_0$ , so that

$$U^{-1}(t, t_0) = U(t_0, t).$$

Finally, we assume that time evolution from  $t_0$  to  $t$  can be viewed as time evolution from  $t_0$  to  $t_1$  and then from  $t_1$  to  $t$ , with  $t_0 < t_1 < t$ , so that

$$U(t, t_1)U(t_1, t_0) = U(t, t_0).$$

The operator  $U$  is called the *time evolution operator*. The principal issue when studying dynamics using quantum mechanics is to uncover the nature of the time evolution operator for the given system. Generally speaking, this is not easy to do if one proceeds in a direct fashion. After all, the time evolution operator has to be a pretty complicated object, given that it knows how to evolve any initial state. A deep principle of physics/mathematics going back to Lie is the following. When considering continuous transformations it is always easier to work with infinitesimal transformations. This is because the finite transformation is completely characterized by the infinitesimal transformation, which is a much simpler structure than the finite transformation. This is one of the reasons that you usually find dynamical evolution (whether in Newtonian dynamics, fluid dynamics, electrodynamics, etc.) expressed in terms of differential equations.

Thus we consider the infinitesimal generator of time evolution, viewed as a continuous unitary transformation. The infinitesimal generator of the unitary time evolution will be an observable called the *Hamiltonian* (in analogy with classical mechanics where the Hamiltonian is the generating function of a canonical transformation corresponding to motion in time). Normally, the Hamiltonian represents the energy, which is conserved provided  $H$  does not depend upon the time. Indeed, just as we defined the momentum as the generator of spatial translations, we *define* the Hamiltonian/energy as the generator of time translations. One subtlety which occurs here, unlike what occurs with spatial translations, is that the Hamiltonian may be time dependent, that is, it may be a family of operators parameterized by the time. This demands a slightly more sophisticated analysis than we used for the spatial translations.