

## Lecture 10

Relevant sections in text: §1.7

**Gaussian state**

Here we consider the important example of a Gaussian state for a particle moving in 1-d. Our treatment is virtually identical to that in the text, but this example is sufficiently instructive to give it again here.

We define this state by giving its components in the position basis, *i.e.*, its wave function:

$$\psi(x) = \langle x|\psi\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{d}}\right) \exp\left(ikx - \frac{x^2}{2d^2}\right).$$

You can check as a good exercise that this wave function is normalized:

$$1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx \langle\psi|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx |\psi(x)|^2.$$

Roughly speaking, the wave function is oscillatory with wavelength  $\frac{2\pi}{k}$  but sitting in a Gaussian envelope centered at the origin. The probability density for position is a Gaussian centered at the origin with width determined by  $d$ . Thus this state represents a particle “localized” near the origin within a statistical uncertainty specified by  $d$ .

Let us make this more precise and exhibit some properties of this state. As an exercise you can check the following results.

$$\langle X\rangle = \langle\psi|X|\psi\rangle = \int_{-\infty}^{\infty} dx x |\psi(x)|^2 = 0.$$

so that the mean location of the particle is at the origin in this state. Next we have

$$\langle X^2\rangle = \langle\psi|X^2|\psi\rangle = \int_{-\infty}^{\infty} dx x^2 |\psi(x)|^2 = \frac{d^2}{2}.$$

Thus the dispersion in position is

$$\langle\Delta X^2\rangle = \frac{d^2}{2},$$

*i.e.*,  $\frac{d}{\sqrt{2}}$  is the standard deviation of the probability distribution for position. Next we have

$$\langle P\rangle = \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} dx \psi^*(x) \frac{\hbar}{i} \frac{d}{dx} \psi(x) = \hbar k,$$

telling us that, on the average, this state has the particle moving with momentum  $\hbar k$ , and

$$\langle P^2\rangle = \langle\psi|P^2|\psi\rangle = \int_{-\infty}^{\infty} dx \psi^*(x) (-\hbar^2) \frac{d^2}{dx^2} \psi(x) = \frac{\hbar^2}{2d^2} + \hbar^2 k^2,$$

so that the momentum uncertainty is

$$\langle \Delta P^2 \rangle = \frac{\hbar^2}{2d^2}.$$

Thus the momentum uncertainty varies reciprocally with  $d$  relative to the position uncertainty. The product of position and momentum uncertainties is as small as allowed by the uncertainty relation. One sometimes calls  $|\psi\rangle$  a *minimum uncertainty state*.

Because the Fourier transform of a Gaussian function is another Gaussian, it happens that the momentum probability distribution is a Gaussian:

$$\begin{aligned} \tilde{\psi}(p) &= \langle p|\psi\rangle = \int_{-\infty}^{\infty} dx \langle p|x\rangle \langle x|\psi\rangle \\ &= \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px} \psi(x) \\ &= \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left\{-\frac{1}{2\hbar^2}(p - \hbar k)^2 d^2\right\}. \end{aligned}$$

You can see that this Gaussian is peaked about the expected momentum value, as it should be, and that its width varies like  $1/d$ , *i.e.*, reciprocal to the position uncertainty, as expected.

In summary, the Gaussian state we have defined corresponds to a particle which (on the average) is moving, and has a Gaussian spread of position and momentum values such that the uncertainty product is minimized. One can use such states to model macroscopic objects. Just put in reasonable values for  $d$  and  $k$  and you will find that the quantum uncertainties are sufficiently small to be negligible.

## Combining systems in quantum mechanics

In our generalization from a particle moving in 1-d to a particle moving in 3-d we tacitly took advantage of a general quantum mechanical scheme for combining systems to make composite systems. In particular, we viewed a particle moving in 3-d as essentially 3 copies of a particle moving in 1-d. We do this sort of thing all the time in physics. For example, if we have a satisfactory kinematic model of a particle and we want to consider a kinematic model for 2 particles, we naturally try to use two copies of the model that we used for one particle. As another example, we have presented a model of a spin 1/2 system, what if we want to describe 2 spin 1/2 systems? Or what if we want to describe a particle with spin 1/2? All these situations require us to know how to combine quantum mechanical models to make composite models. Here we outline the scheme using our generalization from a 1-d particle to a 3-d particle as illustration. Later we will have occasion to use this construction again.

### The tensor product.

Suppose we wish to combine two systems described by Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , with observables represented by linear operators  $A_1, B_1, \dots$  and  $A_2, B_2, \dots$ , which are, of course, only defined on their respective Hilbert spaces. We can build up a Hilbert space for the combined system as follows. We first consider the set consisting of all ordered pairs of vectors from each Hilbert space. Typical elements are denoted

$$|\psi, \chi\rangle := |\psi\rangle \otimes |\chi\rangle, \quad |\psi\rangle \in \mathcal{H}_1, \quad |\chi\rangle \in \mathcal{H}_2.$$

These vectors are called *product vectors*. Physically, a product vector  $|\psi\rangle \otimes |\chi\rangle$  is a state of the composite system in which system 1 is in state  $|\psi\rangle$  and system 2 is in state  $|\chi\rangle$ .

Product vectors admit a natural scalar multiplication inherited from that existing on  $\mathcal{H}_1$  and  $\mathcal{H}_2$ :

$$c|\psi, \chi\rangle := (c|\psi\rangle) \otimes |\chi\rangle \equiv |\psi\rangle \otimes (c|\chi\rangle).$$

It is possible to define a notion of addition on special product vectors. We have

$$|\alpha\rangle \otimes |\beta\rangle + |\gamma\rangle \otimes |\beta\rangle := (|\alpha\rangle + |\gamma\rangle) \otimes |\beta\rangle,$$

and

$$|\alpha\rangle \otimes |\beta\rangle + |\alpha\rangle \otimes |\delta\rangle = |\alpha\rangle \otimes (|\beta\rangle + |\delta\rangle).$$

Obviously, this does not define an addition operation on all product vectors. For additions that are not of the above special form we simply define

$$|\alpha\rangle \otimes |\beta\rangle + |\gamma\rangle \otimes |\delta\rangle,$$

to be a new element of our set. We then simply take the set of all formal linear combinations of product vectors to be the set used in the new, composite Hilbert space. This set is a vector space, called the *tensor product* of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , and is denoted by  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Our text uses the terminology “direct product” for this construction.

A basis for the tensor product can be obtained by taking all product vectors built from a basis for each of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  (exercise). If  $|e_i\rangle$  are a basis for  $\mathcal{H}_1$  and  $|f_j\rangle$  are a basis for  $\mathcal{H}_2$ , then every  $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$  can be written as

$$|\psi\rangle = \sum_{i,j} a_{ij} |e_i\rangle \otimes |f_j\rangle.$$

Note that not every basis for the tensor product is going to be built from product vectors. Note also that the dimension of the product space is the product of the dimensions of the individual Hilbert spaces. In other words, if  $n_1$  is the dimension of  $\mathcal{H}_1$  and  $n_2$  is the dimension of  $\mathcal{H}_2$ , then  $\mathcal{H}_1 \otimes \mathcal{H}_2$  has dimension  $n_1 n_2$ . To see this, simply note that there

are  $n_1 n_2$  scalars in the array of numbers  $a_{ij}$  in the above expansion of an arbitrary vector in the product basis.

The tensor product space is to be a Hilbert space. The scalar product is defined as follows. For product vectors we have

$$\langle \alpha, \beta | \gamma, \delta \rangle = \langle \alpha | \gamma \rangle \langle \beta | \delta \rangle.$$

For general vectors we expand them in a basis of product vectors and then define the scalar product via the usual linearity/anti-linearity of the scalar product (exercise).

Let us note that, since the space of states must be a vector space this forces us to consider as state vectors all possible (normalizable) linear combinations of the product state vectors. Such linear combinations will not be interpretable as saying that each subsystem is in a definite state. Rather, in such linear combinations each subsystem will have various probabilities to be in various states. Moreover, there will exist correlations between the probability distributions for each subsystem. This is the essence of quantum *entanglement*; it is responsible for very striking physical behavior of composite systems compared to their “classical” analogs. Hopefully we will have a little time to discuss this further at the end of the semester. As an example, consider a system consisting of 2 spin 1/2 systems (e.g, an electron and a positron with all degrees of freedom except for spin ignored). A (product) basis for the Hilbert space of states could be the four combinations of  $|\pm\rangle \otimes |\pm\rangle$ . Where the first vector denotes  $S_z$  eigenvectors of particle 1 and the second denotes  $S_z$  eigenvectors of particle 2. These vectors represent states in which  $S_z$  for each particle is known with certainty. A vector such as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |+\rangle + |-\rangle \otimes |-\rangle)$$

represents a state in which neither particle has statistically certain values for  $S_z$ .

You can easily check that all the constructions above can be implemented in an identical fashion for the space of bras. In particular we have

$$\langle \alpha | \otimes \langle \beta | = (\langle \alpha | \otimes \langle \beta |)^\dagger.$$

As an example, let us consider a particle moving in 2-d as built from a product of 2 particles (labeled by  $x$  and  $y$ ) moving in 1-d. A basis for the particle moving in  $x$  is  $|x\rangle$  and a basis for the particle moving in  $y$  is  $|y\rangle$ . The corresponding product basis is then  $|x\rangle \otimes |y\rangle$ . A general vector in the product space can be written as

$$|\psi\rangle = \int d^2x (|x\rangle \otimes |y\rangle) (\langle x | \otimes \langle y |) |\psi\rangle.$$

We define

$$\psi(x, y) = (\langle x| \otimes \langle y|) |\psi\rangle.$$

You see that vectors are characterized by their position wave functions which are complex-valued functions of two variables. These wave functions must be square-integrable as you can see by setting

$$|\psi\rangle = \int dx dy \psi(x, y) |x\rangle \otimes |y\rangle,$$
$$\langle\psi| = \int dx' dy' \psi^*(x', y') \langle x'| \otimes \langle y'|,$$

and computing (exercise)

$$\langle\psi|\psi\rangle = \int d^2x |\psi(x, y)|^2.$$

Of course, with unit normalization, one can interpret  $|\psi(x, y)|^2$  to be the probability density for finding position at  $(x, y)$ .