Lecture 8

Relevant sections in text: §1.6

Momentum

How shall we view momentum in quantum mechanics? Should it be "mass times velocity", or what? Our approach to the definition of momentum in quantum mechanics will rely on a rather fundamental understanding of what is "momentum". To motivate our definition, let me remind you that the principal utility of the quantity called "momentum" is due to its conservation for a closed system. One can then understand the motion of interacting systems via an "exchange of momentum". Next, recall the intimate connection between symmetries of laws of physics and corresponding conservation laws. In particular, symmetry under spatial translations corresponds to conservation of linear momentum. In the Hamiltonian formulation of the classical limit of mechanics this correspondence becomes especially transparent when it is seen that the momentum is the infinitesimal generator of translations, viewed as canonical transformations. In the Hamiltonian framework, the conservation of momentum is identified with the statement that the Hamiltonian is translationally invariant, that is, is unchanged by the canonical transformation generated by the momentum. We shall see that this same logic applies in quantum mechanics. Indeed, nowadays momentum is mathematically identified – by definition – as the generator of translations. Let us see how all this works.

Having defined the position (generalized) eigenvectors, which represent (idealized) states in which the position of the particle is known with certainty, we can define a *translation operator* T_a via

$$T_a|x\rangle = |x + a\rangle.$$

Since the $|x\rangle$ span the Hilbert space, this defines the operator. Note that (exercise)

$$T_a T_b = T_{a+b}, \quad T_0 = I.$$

Physically, we interpret this operator as taking the state of the particle and transforming it to the state in which the particle has been moved by an amount a in the positive x direction. Since we demand that this operator map states into states, we must require that they stay of unit length:

$$1 = \langle\psi|\psi\rangle = \langle\psi|T_a^\dagger T_a|\psi\rangle.$$

It can be shown that this requirement (for all vectors) forces (for all a, which we suppress here)

$$T^\dagger T = TT^\dagger = I, \Longleftrightarrow T^\dagger = T^{-1}.$$

We say that the operator T satisfying this last set of relations is *unitary*. Note that a unitary operator preserves all scalar products (exercise).

Note that for position wave functions we have (good exercise)

$$T_a\psi(x) = \langle x|T_a|\psi\rangle = \langle x-a|\psi\rangle = \psi(x-a).$$

So, moving the system to the right by an amount a shifts the argument of the wave function to the *left*. To see that this makes sense, consider for example a particle with a Gaussian wave function (exercise). The unitarity of T_a is expressed in the position wave function representation via (exercise)

$$\int_{-\infty}^{\infty} |\psi(x-a)|^2 = \int_{-\infty}^{\infty} |\psi(x)|^2 = 1.$$

As we mentioned above, momentum is identified with the infinitesimal generator of translations. Thus, consider an infinitesimal translation, T_ϵ, $\epsilon << 1$. We assume that T_a is continuous in a so that we may expand the operator in a Taylor series

$$T_\epsilon = I - \epsilon\frac{i}{\hbar}P + \mathcal{O}(\epsilon^2).$$

The mathematical definition of the Taylor series of an operator needs a fair amount of discussion, which we will suppress. For our purposes you can just interpret the expansion as meaning that any matrix elements of the operator can be so expanded. The factor of $-\frac{i}{\hbar}$ has been inserted for later convenience. Here P is a linear operator, called the *infinitesimal generator of translations*.* The unitarity and continuity of T implies that P is self-adjoint. Indeed, considering the $\mathcal{O}(\epsilon)$ terms you can easily see (exercise)

$$I = T_\epsilon^\dagger T_\epsilon = (I + \epsilon\frac{i}{\hbar}P^\dagger + \mathcal{O}(\epsilon^2))(I - \epsilon\frac{i}{\hbar}P + \mathcal{O}(\epsilon^2)) \Longrightarrow P = P^\dagger.$$

In fact, the self-adjointness of P is also sufficient for T to be unitary. This can be seen by representing T_a as an infinite product of infinitesimal transformations:

$$T_a = \lim_{N\to\infty}(I - \frac{i}{\hbar}\frac{a}{N}P)^N = e^{-\frac{i}{\hbar}aP}.$$

It is not hard to check that any operator of the form e^{iA} with $A^\dagger = A$ is unitary (exercise). Thus P represents an observable, which we identify with the momentum of the particle.

The canonical commutation relations

We now consider the commutation relation between position and momentum. We can derive this relation by studying the commutator between position and translation operators

* Technically, the infinitesimal generator is $-\frac{i}{\hbar}P$, but it is a convenient abuse of terminology – and it is customary – to call P the infinitesimal generator.

and then considering the limit in which the translation is infinitesimal. Check the following computations as an exercise.

$$XT_\epsilon|x\rangle = (x+\epsilon)|x+\epsilon\rangle.$$

$$T_\epsilon X|x\rangle = x|x+\epsilon\rangle.$$

Subtracting these two relations, taking account of the definition of momentum, and working consistently to first-order in ϵ we have (exercise)

$$X(-\frac{i}{\hbar}P)|x\rangle - (-\frac{i}{\hbar}P)X|x\rangle = |x\rangle.$$

This implies – keep in mind that $|x\rangle$ is a (generalized) basis –

$$[X, P] = i\hbar I.$$

This relation, along with the (trivial) relations

$$[X, X] = 0 = [P, P],$$

constitute the *canonical commutation relations* for a particle moving in one dimension.

Note that these commutation relations show us that position and momentum are *incompatible* observables. They thus satisfy an uncertainty relation:

$$\langle \Delta X^2\rangle\langle\Delta P^2\rangle \geq \frac{1}{4}\hbar^2.$$

This is the celebrated position-momentum uncertainty relation. It shows that if you try to construct a state with a very small dispersion in X (or P) then the dispersion in P (or X) must become large. Note also that the uncertainty relation shows the dispersion in position or and/or momentum can never vanish! However, either of them can be made arbitrarily small provided the other observable has a sufficiently large dispersion.

Momentum as a derivative

It is now easy to see how the traditional representation arises in which momentum is a derivative operator acting on position wave functions. Consider the change in a position wave function under an infinitesimal translation. We have

$$T_\epsilon\psi(x) = (I - \frac{i}{\hbar}P)\psi(x) = \psi(x-\epsilon) = \psi(x) - \epsilon\frac{d\psi(x)}{dx} + \mathcal{O}(\epsilon^2).$$

Comparing terms of order ϵ we see that

$$P\psi(x) \equiv \langle x|P|\psi\rangle = \frac{\hbar}{i}\frac{d\psi(x)}{dx}.$$

Of course, you can now easily verify the position wave function representation of the canonical commutation relations:

$$[X, P]\psi(x) = i\hbar\psi(x).$$