

*Lecture 5**Relevant sections in text: §1.2–1.4***An alternate third postulate**

Here is an equivalent statement of the third postulate.*

Alternate Third Postulate

Let A be a Hermitian operator with an ON basis of eigenvectors $|i\rangle$:

$$A|i\rangle = a_i|i\rangle.$$

The only possible outcome of a measurement of the observable represented by A is one of its eigenvalues a_j . The probability for getting a_j in a state $|\psi\rangle$ is

$$Prob(A = a_j) = \sum_{i=1}^D |\langle i, a_j | \psi \rangle|^2,$$

where the sum ranges over the D -dimensional space of eigenvectors $|i, a_j\rangle, i = 1, \dots, D$, with the given eigenvalue a_j . Note that if the eigenvalue a_k is non-degenerate then

$$Prob(A = a_j) = |\langle j | \psi \rangle|^2.$$

Note also that $\langle j | \psi \rangle$ is the component of $|\psi\rangle$ along $|j\rangle$.

Let us prove the probability formula in the case where there is no degeneracy. Allowing for degeneracy is no problem; think about it as an exercise. Let $f(x)$ be the characteristic function of a_j . We have (exercise)

$$f(A) = |j\rangle\langle j|.$$

Then, in the state $|\psi\rangle$ we have

$$Prob(A = a_j) = \langle f(A) \rangle = \langle \psi | (|j\rangle\langle j|) | \psi \rangle = |\langle j | \psi \rangle|^2.$$

Because the state vectors have unit norm, we have

$$1 = \langle \psi | \psi \rangle = \sum_j \langle \psi | j \rangle \langle j | \psi \rangle = \sum_j |\langle j | \psi \rangle|^2,$$

* For now we state it in a form that is applicable to finite-dimensional Hilbert spaces. We will show how to generalize it to infinite-dimensional ones (such as occur with a “particle”) in a little while.

that is, the probabilities $Prob(A = a_j)$ add up to unity when summed over all the eigenvalues. This indicates that the probability vanishes for finding a value for A which is *not* one of its eigenvalues.

We can write the expectation value of an observable so that the probabilities feature explicitly. As usual, we have

$$A|i\rangle = a_i|i\rangle, \quad \langle i|j\rangle = \delta_{ij}.$$

If a system is in the state $|\psi\rangle$, we compute (by inserting the identity operator twice – good exercise)

$$\begin{aligned} \langle A \rangle &= \langle \psi|A|\psi \rangle \\ &= \sum_{ij} \langle \psi|i\rangle \langle i|A|j\rangle \langle j|\psi \rangle \\ &= \sum_i a_i |\langle i|\psi\rangle|^2. \end{aligned}$$

We see that the expectation value is just the sum of all possible outcomes weighted by the probability for each outcome — just as it should be!

Note the special role played by (normalized) eigenvectors: If the state is an eigenvector of some observable A with eigenvalue a , then the probability for getting a is unity. Eigenstates are states where you know one (or more) observables with certainty.

Here's a good exercise for you: Show that every state vector is an eigenvector of some Hermitian operator. Therefore, at least in principle, every state can be determined by a suitable measurement.

The third postulate makes very specific predictions about what are the possible outcomes of a measurement of a given observable. In essence, all the physical output of quantum mechanics appears via this postulate. I have to emphasize that the predictions are always of a probabilistic nature. (Of course, some probabilities are 1 or 0, so this does *not* mean that one can never make any statements with certainty.)

Let us apply this new form of the third postulate to a couple of simple examples coming from the spin 1/2 system. Consider the state

$$|\psi\rangle = |S_y, +\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle).$$

What is the probability for finding, say, S_y to have the value $-\hbar/2$? Since

$$|S_y, -\rangle = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)$$

is orthogonal to $|S_y, +\rangle$,

$$\langle S_y, -|S_y, +\rangle = 0,$$

this probability is zero. In the same state, what is the probability for finding S_z to have the value $\hbar/2$? We have

$$|\langle S_z, + | S_y, + \rangle|^2 = \left| \frac{i}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

And so forth ...

Stern-Gerlach revisited

Let us now *derive* from our model of the spin 1/2 system some of the salient features of the Stern-Gerlach experiment. View our beam of spin 1/2 system as a large number of identical systems. When we pass the beam through an apparatus SG_z , and keep only the spin up beam, we have a large collection of particles, all in the state $|S_z, +\rangle$. If we pass the beam through another SG_z apparatus, we find all the particles are still spin up. In our model, this result is simply the statement that the probability for finding $S_z = \hbar/2$ in a the state $|\psi\rangle = |S_z, +\rangle$ is unity:

$$|\langle S_z, + | \psi \rangle|^2 = \langle S_z, + | S_z, + \rangle|^2 = 1.$$

Now consider passing the filtered beam through an SG_x apparatus. We know that the beam splits in half. In our model this is seen by computing the probability distribution for S_x when $|\psi\rangle = |S_z, +\rangle$. We get

$$|\langle S_x, \pm | \psi \rangle|^2 = |\langle S_x, \pm | S_z, + \rangle|^2 = \frac{1}{2}.$$

Now we come to a crucial aspect of the use of our model to describe nature. Suppose we take all the particles that had $S_x = \hbar/2$ and pass them through an SG_z apparatus, what does our model predict? The key observation is that the beam of particles that are going to be passed through SG_z are *now* all in the state

$$|\psi\rangle = |S_x, +\rangle = \frac{1}{\sqrt{2}}(|S_z, +\rangle + |S_z, -\rangle).$$

The measurement/preparation/filtering process using SG_x has determined a new state vector for the system! To verify this, we just pass the SG_x -filtered beam through another SG_x apparatus and see that the particles have $S_x = \hbar/2$ with probability unity. This means that they are in the corresponding eigenstate, $|S_x, +\rangle$. So, when we pass this beam through SG_z we find the 50-50 probability distribution:

$$|\langle \pm | \psi \rangle|^2 = |\langle \pm | S_x, + \rangle|^2 = \frac{1}{2}.$$

We now want to spend a little time understanding this sort of result in general.

Compatible and Incompatible Observables

Let us generalize the unusual result found in the Stern-Gerlach experiment as follows. Suppose we have two observables A and B . If we measure A we will get one of its eigenvalues, say, a . For simplicity (only) we assume that this eigenvalue is non-degenerate. The state of the system is now the eigenvector $|a\rangle$, where

$$A|a\rangle = a|a\rangle.$$

If we then measure the observable B we get an eigenvalue, say b , and the system is in the state $|b\rangle$, where

$$B|b\rangle = b|b\rangle$$

(for simplicity, we also assume that b is a non-degenerate eigenvalue). The probability for getting a is now

$$|\langle a|\psi\rangle|^2 = |\langle a|b\rangle|^2$$

Unless there is a real number c such that $|a\rangle = e^{ic}|b\rangle$ this probability is less than unity.* When do we have $|a\rangle = e^{ic}|b\rangle$? This condition means that A and B have a common eigenvector. (Note that there are no common eigenvectors for the spin operators – exercise.) As long as A and B do not have all of their eigenvectors in common, there will be situations where knowing one of the observables with certainty will preclude knowing the other with certainty. Such observables are called *incompatible*. Under what conditions will two observables be *compatible*? Evidently, compatible observables are represented by operators that have a common (basis) set of eigenvectors so that the argument just given above cannot apply. This implies that the observables must *commute*,

$$[A, B] \equiv AB - BA = 0.$$

To see this, let us denote by $|a, b\rangle$ the common set of eigenvectors, which form an ON basis:

$$A|a, b\rangle = a|a, b\rangle, \quad B|a, b\rangle = b|a, b\rangle, \quad \langle a, b|a', b'\rangle = \delta_{aa'}\delta_{bb'}.$$

We have

$$AB|a, b\rangle = bA|a, b\rangle = ba|a, b\rangle = aB|a, b\rangle = BA|a, b\rangle.$$

Since the operator $[A, B]$ maps each element of a basis to the zero vector, and since every vector can be expanded in this basis, we conclude that $[A, B]$ must be the zero operator.

* To see this, use the Schwarz inequality – exercise:

$$|\langle a|b\rangle|^2 \leq \langle a|a\rangle\langle b|b\rangle, \quad \text{equality iff } |a\rangle = (\text{const.})|b\rangle.$$

So, if a pair of Hermitian operators have a common basis of eigenvectors then they must commute. It is a fundamental result from linear algebra (for finite dimensional vector spaces) that the converse also holds: If two Hermitian operators commute, then they admit a common, ON basis of eigenvectors. Thus compatible observables are represented by commuting operators. Physically, it is possible to have states in which the values of compatible observables are determined with certainty – unlike different components of the spin. By the same token, incompatible observables are represented by operators that do not commute. For incompatible observables there will exist states in which their values cannot be determined with statistical certainty. We see then that the unusual feature of nature embodied in the Stern-Gerlach experiment is encoded in the *algebraic structure* associated to the observables by the commutator of the corresponding linear operators.