ECE 6010
Lecture 8 – Random Processes in Linear Systems

Continuous time systems

Recall: A signal X_t through a linear system produces an output

$$Y_t = \int_{-\infty}^{\infty} h(t,s) X_s ds.$$

- The system is **causal** if $h(t,s) = 0$ for $t < s$. In this case,

$$Y_t = \int_{-\infty}^{t} h(t,s) X_s ds.$$

- The system is **time invariant** if $h(t,s) = h(t-s, 0) \triangleq h(t-s)$ for all t and s. In this case, Y_t is obtained by convolution:

$$Y_t = \int_{\infty}^{t} h(t-s) X_s ds = h * X.$$

  In this case, we can do analysis using Fourier transforms:

$$Y(\omega) = H(\omega) X(\omega)$$

Suppose the input function is a random process instead of a deterministic signal. How can we characterize the output function?

Let

$$Y_t = \int_{-\infty}^{\infty} h(t,s) X_s \, ds$$

The integral is to be interpreted in in the mean-square sense. Properties:

- $\int_{-\infty}^{\infty} h(t,s) X_s \, ds$ exists $\Leftrightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t,s) h(t,q) R_X(s,q) \, ds dq < \infty$.
- $\mu_Y(t) = \int_{-\infty}^{\infty} h(t,\tau) \mu_X(\tau) d\tau$. That is, the mean of the output is the response of the mean of the input:

$$E[Y_t] = E\left[\int_{-\infty}^{\infty} h(t,\tau) X_\tau \, d1t\right] = \int_{-\infty}^{\infty} h(t,\tau) E[X_t] \, d\tau$$

- $R_{XY}(t,s) = E[Y_t X_s] = \int_{-\infty}^{\infty} h(t,\tau) R_X(\tau, s) \, d\tau$. That is, the correlation between the input and the output is the response to the autocorrelation with respect to the 1st input.
- $R_Y(t,s) = E[Y_t Y_s] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t,\sigma) R_X(\sigma,\tau) h(s,\tau) \, d\sigma d\tau$.
- If $\{X_t, t \in \mathbb{R}\}$ is W.S.S. and h is tine-invariant. Then $\{Y_t\}$ is also W.S.S., and $\{X_t\}$ and $\{Y_t\}$ are jointly W.S.S. and the following hold:

  1. $\mu_y = \mu_x \int_{-\infty}^{\infty} h(t) \, dt$
  2. $R_{YX}(\tau) = \int_{-\infty}^{\infty} h(\tau - \sigma) R_X(\sigma) \, d\sigma = (h * R_X)(\tau)$.
  3. $R_Y(\tau) = \int_{-\infty}^{\infty} h(\sigma - \tau) R_{YX}(\sigma) d\sigma = (\hat{h} * R_{YX})(\tau) = (\hat{h} * H * R_X)(\tau)$, where $\hat{h}(t) = h(-t)$.

4. $S_{YX}(\omega) = H(\omega)S_X(\omega)$
5. $S_Y(\omega) = H^*(\omega)H(\omega)S_X(\omega) = |H(\omega)|^2 S_X(\omega)$. The quantity $|H(\omega)|^2$ is sometimes called the power transfer function.

Example 1 Suppose we have an RC circuit (C in parallel). Then

$$Y_t = \alpha \int_{-\infty}^{t} e^{-\alpha(t-\tau)} X_\tau d\tau,$$

where $\alpha = 1/RC$. Then

$$H(\omega) = \frac{\alpha}{\alpha + i\omega}$$

$$h(t) = \alpha e^{-\alpha t} u(t)$$

If $\mu_x(t) = 0$, $R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$ (that is, X_t is wide-sense Markov), we have

$$S_X(\omega) = \frac{2\beta\sigma^2}{\omega^2 + \beta^2}$$

Then

$$\mu_y(t) = 0$$

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \frac{\alpha^2}{\alpha^2 + \omega^2} \frac{2\beta\sigma^2}{\beta^2 + \omega^2}$$

We can find $R_Y(\tau)$ by an inverse F.T.:

$$R_Y(\tau) = \frac{\alpha^2\sigma^2}{\beta^2 - \alpha^2} \left(\frac{\beta}{\alpha} e^{-\alpha|\tau|} - e^{-\beta|\tau|} \right)$$

□

Discrete-time filters

Let $T = \mathbb{Z}$. The output of a discrete-time system is

$$Y_k = \sum_{l=-\infty}^{\infty} h_{k,l} X_l$$

Causal: $h_{k.l} = 0$ for $k < l$. Time-invariant: $h_{k,l} = h_{k-l,0} = h_{k-l}$ for all $k, l \in \mathbb{Z}$. Then

$$Y_k = \sum_{l=-\infty}^{\infty} h_{k-l} X_l = (h * X)_k.$$

We can transform in the time-invariant case using a Z-transform,

$$Y(z) = H(z)X(z),$$

or

$$Y_k = \frac{1}{2\pi i} \oint Y(z) z^{k-1}\, dz.$$

We also deal with the discrete-time Fourier transform, obtained by evaluating on the unit circle $z = e^{i\omega}$. We write

$$H(z)|_{z=e^{i\omega}} = H(\omega).$$

(This is an abuse of notation, but makes the notation consistent with continuous time.)

For a random process, we still have

$$Y_k = \sum_{l=-\infty}^{\infty} h_{k-l} X_l = (h * X)_k,$$

but we interpret this in a m.s. sense:

$$\lim_{m,n\to\infty} E[(Y_k - \sum_{l=-m}^{n} h_{k,l} X_l)^2] = 0.$$

Properties:

1. Y_k exists $\Leftrightarrow \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h_{k,l} R_X(l,m) h_{k,m} < \infty$.
2. $\mu_Y(k) = \sum_{l=-\infty}^{\infty} h_{k,l} \mu_x(l)$.
3. $R_{YX}(k,n) = \sum_{l=-\infty}^{\infty} h_{k,l} R_X(l,n)$
4. $R_Y(k,n) = \sum_{l=-\infty}^{\infty} h_{n,l} R_{YX}(k,l) = \sum_l \sum_m h_{n,l} h_{k,m} R_X(m,l)$
5. If $\{X_k\}$ is W.S.S. and $\{h_{k,l}\}$ is time-invariant, then $\{Y_k\}$ is W.S.S. and $\{X_k\}$,$\{Y_k\}$ are jointly W.S.S. Then:

   (a) $\mu_y = \mu_x \sum_k h_k$

   (b) $R_{YX}(k) = (h * R_X)(k)$

   (c) $R_Y(k) = (\hat{h} * X * h)(k)$, where $\hat{h}(k) = h(-k)$.

   These properties can be expressed in the Z-transform domain:

   (a) $\mu_y = \mu_x H(z)|_{z=1}$.

   (b) $S_{YX}(z) = \sum_k R_{YX}(k) z^{-k} = H(Z) S_X(z)$

   (c) $S_Y(z) = \sum_k R_{YX}(z) z^{-k} = H(z^{-1}) H(z) S_X(z)$.

   These can be further expressed on the unit circle as:

   (a) $\mu_y = \mu_x H(\omega)|_{\omega=0}$.

   (b) $S_{YX}(\omega) = H(\omega) S_X(\omega)$

   (c) $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$.

Example 2 Consider the feedback system with

$$Y_k = X_k + \alpha Y_{k-1}$$

with $|\alpha| < 1$. We have

$$h_k = \alpha^k u[k]$$

and

$$H(z) = \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}.$$

so $H(1/z) = 1/(1-\alpha z)$.

Suppose $\mu_X(k) = 0$ and

$$R_X(k) = \begin{cases} N_0/2 & k = 0 \\ 0 & \text{else.} \end{cases}$$

This is discrete-time **white** noise. Then

$$S_X(z) = \frac{N_0}{2}$$

$$S_Y(z) = H(z)H(1/z)S_X(z) = \frac{N_0/2}{(1-\alpha z)(1-\alpha z^{-1})}.$$

Computing the inverse Z-transform we find

$$R_Y(k) = \frac{N_0}{2(1-\alpha^2)}\alpha^{|k|}.$$

This gives us a **discrete-time wide sense Markov process.**

Note that for the white noise signal,

$$R_Y(k,l) = \sum_n \sum_m h(k,n)h(l,m)\underbrace{R_X(m,n)}_{N_0/2\delta_{n,m}} = \frac{N_0}{2}\sum_n h(k,n)h(l,n).$$

For time-invariant systems,

$$S_Y(\omega) = \frac{N_0}{2}|H(\omega)|^2.$$

□

White Noise

We have seen that for a discrete-time signal we can create a "white" noise with the properties

$$\mu_X(k) = 0 \qquad R_X(k,l) = \frac{N_0}{2}\delta_{k,l}.$$

For continuous time random processes, we say that it is white noise if

$$\mu_X(t) = 0$$

$$R_X(t,s) = \frac{N_0}{2}\delta(t-s).$$

This is thus W.S.S. However, since we are dealing with the Dirac δ function, we do not have a second order random process. Because of this, such a process cannot be said to exist in a physical sense. Nevertheless, it is a very important and practical model for use in conjunction with linear systems.

Using a white noise process is fine as long as it is input to a linear system (which will integrate the process, thus smoothing it out).

Note that if the process is Gaussian and white, then the output is also Gaussian. In this case, knowing the first and second distributions (e.g., mean and correlation functions) is sufficient to entirely characterize the f.d.d.s of the joint processes.

We mentioned earlier the **Wiener** random process. Let us consider one now having with zero mean:

$$R_W(t,s) = \sigma^2 \min(t,s) + \mu^2 ts = \sigma^2 \min(t,s)$$

We can approximate the white random process using differences,

$$\{W_{t+\Delta t} - W_t\}.$$

In the limit, we can take the *derivative* of the Wiener process to obtain a white noise process.

What is the autocorrelation function of the derivative of the Wiener process?

$$\frac{\partial R_W(t,s)}{\partial t} = \begin{cases} \sigma^2 & t < s \\ 0 & t > s. \end{cases}$$

$$\frac{\partial^2 R_W(t,s)}{\partial s \partial t} = \sigma^2 \delta(t-s).$$

So we get a W.S.S. white random noise process. Strictly speaking, however, the derivative does not exist in the m.s. sense.

We always employ white noise r.p. in the context of an integration operation (e.g., running through a system).