**ECE 6010**

**Lecture 7 – Analytical Properties of Random Processes**

Let $X_t$ be a function of time, and let $h(t)$ be the impulse response of a (continuous-time) linear time invariant system. If $X_t$ is the input to the system, then the output is

$$Y_t = X_t * h(t) = \int_{-\infty}^{\infty} h(t-\tau) X_\tau \, d\tau.$$

We will consider such operations when $X_t$ is a random process. This will require us to develop some additional analytic properties of random processes.

# Continuity

Let $T = (a, b)$. Recall that a function $f : T \to \mathbb{R}$ is **continuous** at $t_0$ if $\lim_{t \to t_0} f(t) = f(t_0)$. $f$ is continuous if it is continuous for every $t_0 \in (a, b)$.

**Definition 1** For a random process $\{X_t, t \in T\}$, we say $X_t$ is **continuous with probability 1** if $P(\{\omega \in \Omega : X_t(\omega), t \in T \text{ is continuous}\}) = 1$. □

Continuity w.p. 1 is usually quite strong, in fact stronger than is typically needed for analysis.

We say that $\{X_t, t \in T\}$ is **continuous in probability** at $t_0$ if $X_t \to X_{t_0}$ (i.p.) as $t \to t_0$. That is,

$$\lim_{t \to t_0} P(|X_t - X_{t_0}| > \epsilon) = 0 \text{ for all } \epsilon > 0.$$

We say that $\{X_t, t \in T\}$ is **mean-square continuous** at $t_0$ if $X_t \to X_{t_0}$ (m.s.) as $t \to t_0$. That is,

$$\lim_{t \to t_0} E[|X_t - X_{t_0}|^2] = 0.$$

An important property: A second-order random process is mean-square continuous at $t = t_0$ if and only if $R_X(t, s)$ is continuous at $t = s = t_0$, that is

$$\lim_{t \to t_0, s \to t_0} R_X(t, s) = R_X(t_0, t_0).$$

**Proof** (If) Suppose $R_X(t, s)$ is continuous at $t = s = t_0$. Then

$$E[|X_t - X_{t_0}|^2] = E[X_t^2] - 2E[X_t X_{t_0}] + E[X_{t_0}^2] = (R_X(t,t) - R_X(t, t_0)) + (R_X(t_0, t_0) - R_X(t, t_0))$$

Continuity of $R_X$ implies that $\lim_{t \to t_0} R_X(t, t) = \lim_{t \to t_0} R_X(t, t_0) = R_X(t_0, t_0)$. So $\lim_{t \to t_0} E[|X_t - X_{t_0}|^2] = 0$.

(Only if) We have

$$\begin{aligned}
|R_X(t,s) - R_X(t_0,t_0)| &= |E[X_t X_s] - E[X_{t_0} X_{t_0}]| \\
&= |E[(X_t - X_{t_0})(X_s - X_{t_0})] + E[(X_t - X_{t_0})X_{t_0}] + E[(X_s - X_{t_0})X_{t_0}]| \\
&\le |(E[X_t - X_{t_0})(X_s - X_{t_0})]| + |E[(X_t - X_{t_0})X_{t_0}]| + |E[(X_s - X_{t_0})X_{t_0}]| \\
&\le \sqrt{E[(X_t - X_{t_0})^2]E[(X_s - X_{t_0})^2]} + \sqrt{E[(X_t - X_{t_0})^2]E[X_{t_0}^2]} + \sqrt{E[(X_s - X_{t_0})^2]E[X_{t_0}]}.
\end{aligned}$$

Since $E[X_t - X_{t_0}] \to 0$ as $t \to t_0$, each term in the sum $\to 0$. Thus $R_X(t, s)$ is continuous at $t = s = t_0$. □

**Example 1** (Homogeneous Poisson counting process). $R_X(t, s) = \lambda \min(t, s) + \lambda^2 ts$. This is continuous at every point at every $t = s = t_0$ for every $t_0 \in \mathbb{R}^+$. So the process is mean-square continuous.

But recall the sample path is a series of jumps: any realization is discontinuous with probability 1. □

**Example 2** Let $X_t$ be a Gaussian random process with

$$R_X(t,s) = \sigma^2 \min(t,s) + \mu^2 ts.$$

This is also mean-square continuous. It can be shown that this process is also continuous with probability 1.

(This process is called the **Wiener process**; we will have more to say about it later.) It models random diffusion or Brownian motion. □

For a W.S.S. process, we have the following: A W.S.S. r.p. $X_t$ is mean-square continuous if and only if $R_X(\tau)$ is continuous at $\tau = 0$. This follows since $R_X(t,s) = R_X(t-s)$, and

$$\lim_{(t,s)\to(t_0,t_0)} R_X(t,s) = \lim_{\tau\to 0} R_X(\tau).$$

# Differentiation

Recall that $f : T \to \mathbb{R}$ is **differentiable** at $t_0 \in T$ if

$$\lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0} = f'(t_0)$$

exists.

Similarly, a r.p. $\{X_t, t \in T\}$ is mean-square differentiable at $t_0$ if

$$X'_{t_0} = \lim_{t\to t_0} \frac{X_t - X_{t_0}}{t - t_0}$$

exists in the mean-square sense, that is,

$$E[\left|X'_{t_0} - \frac{X_t - X_{t_0}}{t - t_0}\right|^2] \to 0 \text{ as } t \to t_0.$$

If $X_t$ is mean-square differentiable at every $t_0 \in T$, then $X'_t$ defines another random process on the underlying probability space $(\omega, \mathcal{F}, P)$.

Suppose $Y_t$ is a second-order random process, and $\lim_{t\to t_0} Y_t = Z$ (m.s.). Then:

1. $E[Z^2] < \infty$;
2. and if $E[X^2] < \infty$ then
$$\lim_{t\to t_0} E[Y_t X] = E[ZX].$$

**Proof**

1. $Z = Y_t + Z - Y_t$. Then since $(a+b)^2 \leq 4a^2 + 4b^2$, we have
$$Z^2 \leq 4Y_t^2 + 4(Z - Y_t)^2$$
and
$$E[Z^2] \leq 4E[Y_t^2] + 4E[(Z - Y_t)^2]$$
But each of these are $< \infty$ for $t$ sufficiently close to $t_0$ (by mean-square convergence.) So $E[Z^2] < \infty$ for all $t$.
2. We have
$$\begin{aligned} 0 &\leq \left|\lim_{t\to t_0} E[Y_t X] - E[ZX]\right|^2 \\ &= |E[(Y_t - Z)X]|^2 \\ &\leq E[(Y_t - Z)^2]E[X^2] \\ &= 0. \end{aligned}$$

□

## Properties of the derivative

Suppose $\{X_t, t \in T\}$ is mean-square differentiable with mean-square derivative $\{X'_t, t \in T\}$. Suppose that $\{X_t\}$ is second order. Then:

1. $\{X'_t\}$ is also second order. (This follows from the first fact above.)
2. $\frac{\partial}{\partial t} R_X(t,s)$ exists, and equals $R_{X',X}(t,s)$.

   **Proof**

$$\begin{aligned} R_{X'X}(t,s) &= E[X'_t X_s] = E[\lim_{q\to t} \frac{X_q - X_t}{q-t} X_s] \\ &= \lim_{q\to t} E[\frac{X_q - X_t}{q-t} X_s] \qquad \text{(from fact 2 above)} \\ &= \lim_{q\to t} \frac{E[X_q X_s] - E[X_t X_s]}{q-t} \\ &= \lim_{q\to t} \frac{R_X(q,s) - R_X(t,s)}{q-t} \\ &= \frac{\partial}{\partial t} R_X(t,s). \end{aligned}$$

   □

3. $\frac{\partial}{\partial s}\frac{\partial}{\partial t} R_X(t,s)$ exists and is equal to $R_{X'}(t,s)$ for all $t, s \in T$.

   **Proof**

$$\begin{aligned} R_{X'}(t,s) &= E[X'_t X'_s] = E[X'_t \lim_{q\to s} \frac{X_q - X_s}{q-s} \\ &= \lim_{q\to s} \frac{E[X'_t X_q] - E[X'_t X_s]}{q-s} \qquad \text{(from fact 2 above)} \\ &= \lim_{q\to s} \frac{R_{X'X}(t,s) - R_{X'X}(t,s)}{q-s} = \lim_{q\to s} \frac{\frac{\partial}{\partial t} R_X(t,s) - \frac{\partial}{\partial t} R_X(t,s)}{q-s} \\ &= \frac{\partial}{\partial s}\frac{\partial}{\partial t} R_X(t,s). \end{aligned}$$

   □

4. $\frac{\partial}{\partial t}\mu_X(t)$ exists and is equal to $\mu_{X'}(t)$.
5. Suppose $T = \mathbb{R}$ and $X_t$ is also W.S.S. Then $\{X'_t\}$ is also W.S.S. Also, $\{X_t\}$ and $\{X'_t\}$ are jointly W.S.S. Also,
   (a) $\mu'_x = 0$.
   (b) $R_{X'X}(\tau) = \frac{d}{d\tau} R_X(\tau)$.
   (c) $R_{X'}(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau)$.

## On the existence of the mean-square derivative

It can be shown that a **sufficient** condition for the existence of $X'_{t_0}$ is the existence of $\frac{\partial^2}{\partial t \partial s} R_X(t,s)$ at $(t,s) = (t_0, t_0)$. A **necessary** condition is the existence and equality of the mixed partials

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial s} R_X(t,s)\right]_{(t=s=t_0)} = \frac{\partial}{\partial s}\left[\frac{\partial}{\partial t} R_X(t,s)\right]_{(t=s=t_0)}.$$

If $X_t$ is W.S.S., then these two conditions are the same. So $X'_{t_0}$ exists if and only if

$$\left|\frac{d^2}{d\tau^2}R_X(\tau)\right|_{\tau=0} < \infty.$$

By a result we will show later, we will find that

$$R_{X'}(0) = -R_X(0)\int_{-\infty}^{\infty}\omega^2 S_X(\omega).$$

So the existence of $X'_t$ for a WSS process $X_t$ is equivalent to the condition that the second moment of the PSD of $X_t$ is finite.

**Example 3** Let $X_t$ be a process having

$$S_X(\omega) = \frac{k}{\omega^2+\beta^2}.$$

In this case,

$$-R_{X'}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2\frac{k}{\omega^2+\beta^2}\,d\omega = \infty,$$

so the process is *not* mean-square differentiable. □

# Integration

For a function $f : T \to \mathbb{R}$, we define a Riemann integral as

$$\lim_{\max|t_i-t_{i-1}|\to 0}\sum_{i=1}^{n} f(\zeta_i)(t_i-t_{i-1}) = \int_a^b f(t)\,dt$$

where $a = t_0 < t_1 < \cdots < t_n = b$ and $z_i \in (t_{i-1}, t_i)$.

Let us define a similar sort of limit for a random process. We will define the limits in the mean-square sense. Then

$$\int_a^b X_t\,dt$$

is the mean-square integral of $X_t$.

Properties of M.S. integrals $\int_a^b X_t\,dt$:

1. The integral exists if and only if

$$\int_a^b\int_a^b R_X(t,s)\,dtds < \infty.$$

**Proof** $\int_a^b X_t\,dt$ exists if and only if

$$\lim_{\max|t_i-t_{i-1}|\to 0,\max|s_i-s_{i-1}|\to 0} E\left[\left(\sum_{i=1}^{n}X_{\zeta_i}(t_i-t_{i-1}) - \sum_{j=1}^{m}X_{\beta_j}(s_j-s_{j-1})\right)^2\right] = 0$$

$$= \lim\sum_{i=1}^{n}\sum_{k=1}^{n}E[X_{\zeta_i}X_{\zeta_k}](t_i-t_{i-1})(t_k-t_{k-1}) - 2\sum_{i=1}^{n}\sum_{j=1}^{m}E[X_{\zeta_i}X_{\beta_j}](t_i-t_{i-1})(s_j-s_{j-1}) +$$
$$\sum_{j=1}^{m}\sum_{l=1}^{m}E[X_{\beta_j}X_{\beta_l}](s_j-s_{j-1})(s_l-s_{l-1})$$

The expectations can be replaced as

$$\lim \sum_{i=1}^{n}\sum_{k=1}^{n} R_X(\zeta_i,\zeta_k)(t_i - t_{i-1})(t_k - t_{k-1}) - 2\sum_{i=1}^{n}\sum_{j=1}^{m} R_X(\zeta_i,\beta_j)(t_i - t_{i-1})(s_j - s_{j-1}) + \sum_{j=1}^{m}\sum_{l=1}^{m} R_X(\beta_j,\beta_l)(s_j - s_{j-1})(s_l - s_{l-1}) = 0$$

The equality to zero is the Cauchy criterion for the Riemann sums defining $\int_a^b \int_a^b R_X(t,s)\,dtds$. Thus, the integral exists implies that $\int_a^b \int_a^b R_X(t,s)\,dtds$ exists. □

2. Assume that $\int_a^b X_t dt$ exists. Then $E[\int_a^b X_t\,dt] = \int_a^b E[X_t]\,dt = \int_a^b \mu_x(t)\,dt$.
3. $E[(\int_a^b X_t\,dt)^2] = \int_a^b \int_a^b R_X(t,s)\,dtds$.