ECE 6010
Lecture 6 – Basic Concepts of Random Processes

Basic definitions and concepts

Definition 1 A **random process** (or **stochastic process** on a probability space $(\Lambda, \mathcal{F}, P)$ is an indexed collection of random variables $\{X_t, t \in T\}$ each defined on $(\Omega, \mathcal{F}, P)$, where T is an indexing set of real numbers. $\square$

- If T is a singleton (one element) then $\{X_t, t \in T\}$ is a r.v.
- If $T = \{t_1, t_2\}$, then $\{X_t, t \in T\}$ is a bivariate r.v.
- If T consists of a finite number of elements, then $\{X_t, t \in T\}$ is a random vector.
- If T is countable, then $\{X_t, t \in T\}$ is a random sequence.

For most applications we think of t as "time." In some cases, T is multidimensional. Then X_t is called a **random field.**

Three interpretations of a r.p.

1. A collection of waveforms that occur randomly. That is, it is defined on some probability space. For each $\omega \in \Omega$ there is a corresponding waveform $\{X_t(\omega), t \in T\}$ as a function of t with ω fixed.

   Think of having a big bag of waveforms. We reach into the bag and pick out a waveform — a function of t — at random.

2. A collection of random variables. In this case, that is, for each fixed $t \in T$, we have a random variable X_t.

3. A real-valued function of two variables $X_t : \Omega \times T \to \mathbb{R}$.

Definition 2 A function $\{X_t(\omega), t \in T\}$ assumed by $\{X_t, t \in T\}$ for a fixed $\omega \in \Omega$ is called a **realization** of the process. (Also known as a **sample function** or **sample path**. $\square$
A realization is just a function. It does not exhibit the randomness.

Definition 3 If T contains a continuum of values (e.g., $T = \mathbb{R}$ or $T = [0, 1]$) then $\{X_t, t \in T\}$ is a **continuous-time random process**. $\square$

Definition 4 If T contains only countably many values (e.g. $T = \mathbb{Z}$, $T = \mathbb{Z}^+$) then $\{X_t, t \in T\}$ is a **discrete-time random process.** $\square$

Definition 5 Let n be a positive integer and $\{X_t, t \in T\}$ a random process. The set of n-dimensional distributions of $\{X_t, t \in T\}$ is a collection of all multivariable distributions of collections $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, where $t_i \in T$.

The set of all n-dimensional distributions for all orders n is called the **set of finite-dimensional distributions** (f.d.d.s) of $\{X_t, t \in T\}$.

$\square$

We will assume this set completely characterizes the statistical distribution of the process.

Definition 6 T is closed under addition if $T_1, T_2 \in T$ implies $T_1 + T_2 \in T$. $\square$

Definition 7 Suppose T is closed under addition. The random process $\{X_t, t \in T\}$ is **stationary to order** k if for all $t_1, t_2, \ldots, t_k \in T$, the distribution of $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h}$ does not depend on h for $h \in T$.

If this is true for all orders k, the r.p. is **(strictly) stationary**. $\square$

Strict stationarity is a fairly strong condition, and we don't necessarily need it always.

Example 1 Stationarity to order 1 means that F_{X_t} is the same for every $t \in T$.

Stationarity to order 2 means that F_{X_t,X_s} depends only on the *difference* between t and s. $\square$

Ergodicity

Assume throughout that $\{X_t\}$ is stationary.

Loosely speaking, a random process $\{X_t\}$ is *ergodic* if time averages are equal to ensemble averages. That is, averages over ω — expectations — are the same as averages over t. That is, ensemble averages are the same as sample averages.

Here is an example: Suppose $\{X_n\}$ is an i.i.d. sequence. The ensemble mean is $\mu = \int X(\omega)P(d\omega)$. The sample mean is

$$\frac{1}{n}\sum_{k=1}^{n} X_k(\omega).$$

By the S.L.L.N. we have

$$\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) \to \mu.$$

This is an example of an ergodic property.

Means and Autocorrelations

Definition 8 The **mean function** of a r.p. $\{X_t, t \in T\}$ is

$$\mu_X(t) = E\{X_t\}, \qquad t \in T.$$

$\square$

Definition 9 The **autocorrelation function** of a r.p. $\{X_t, t \in T\}$ is

$$R_X(t,s) = E[X_t X_s], \qquad t,s, \in T.$$

$\square$

Definition 10 A random process is **second order** if $E[X_t^2] < \infty$ for all $t \in T$. $\square$

For a second order r.p., $|\mu_x(t)| < \infty$ and $|R_X(t,s)| < \infty$ for all $t,s, \in T$.

Properties of Autocorrelation functions

1. $R_X(t,t) = E[X_t^2]$. (This is the second moment)
2. $|R_X(t,s)|^2 \leq R_X(t,t)R_X(s,s)$ (Schwartz inequality)
3. $R_X(t,s) = R_X(s,t)$ (symmetric)

Wide-sense stationarity

Definition 11 Let T be closed under addition. A second order random process $\{X_t, t \in T\}$ is said to be **wide-sense stationary** (WSS) if $\mu_x(t)$ is a constant, $\mu_x(t) = \mu_x$ and $R_X(t+h,t)$ depends only on h for all $t, h \in T$. □

Since $R_X(t+h,t)$ depends only on h, we write (by an "abuse of notation")

$$R_X(t+h,t) \equiv R_X(h)$$

Thus,

$$R_X(t,s) = R_X(t-s,0) = R_X(t-s)$$

If a random process is second order and strictly stationary, it must also be WSS. On the other hand, if a process is WSS, it is not necessarily strictly stationary.

Definition 12 The **autocovariance function** of a random process $\{X_t, t \in T\}$ is

$$C_X(t,s) = \operatorname{cov}(X_t, X_s), \qquad t, s \in T.$$

□

We say a process is **covariant stationary** if $C_X(t,s)$ depends only on $t-s$, or, equivalently, $C_X(t+h,t)$ depends only on h.

Properties of R_X for WSS r.p.s

1. $R_X(0) = E[X_t^2]$ (independent of t)
2. $|R_X(\tau)| \leq \sqrt{E[X_{t+\tau}^2]E[X_t^2]} = R_X(0)$.
3. $R_X(\tau) = R_X(-\tau)$ (even function)
4. A defining property of these functions:

   $$\sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_k \alpha_s^* R_X(t_k - t_l) \geq 0$$

   for all $t_1, \ldots, t_n \in T$ and all $\alpha_1, \ldots, \alpha_n$ and for all $n \in \mathbb{Z}^+$.

   Any function with this property is a nonnegative definite function.

Before proceeding with more properties, a few examples.

A Sinusoidal Process

Let $T = \mathbb{R}$. Assume A and θ are independent r.v.s, with $E[A^2] < \infty$ and $\theta \sim \mathcal{U}(-\pi, \pi)$. Define $\{X_t, t \in T\}$ by

$$X_t = A\sin(\omega_0 t + \theta)$$

where ω_0 is a known constant.

A typical realization is a sinusoid.

This is an example of a **deterministic** random process, that is, a random process determined by random parameters.

$$\begin{aligned}\mu_X(t) &= E[A\sin(\omega_0 t + \theta)] = E[A]E[\sin(\omega_0 t + \theta)] \\ &= E[A]\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(\omega_0 t + \theta) d\theta = 0.\end{aligned}$$

We observe that this process is ergodic in the mean — a time average is equal to the ensemble average.

$$\begin{aligned}
R_X(t,s) &= E[X_t X_s] = E[A^2 \sin(\omega_0 t + \theta)\sin(\omega_0 s + \theta)] \\
&= E[A^2]E[\sin(\omega_0 t + \theta)\sin(\omega_0 s + \theta)] \\
&= E[A^2]\frac{1}{2\cdot 2\pi}\int_{-\pi}^{\pi} \cos(\omega_0(t-s)) + \cos(\omega_0(t+s) + 2\theta)\, d\theta \\
&= E[A^2]\frac{1}{2}\cos(\omega_0(t-s)).
\end{aligned}$$

We observe that $R_X(t,s)$ depends only on the time difference $t-s$. Hence, the r.p. is WSS. Let $\tau = t-s$. We can write

$$R_X(\tau) = E[A^2]\frac{\cos(\omega_0\tau)}{2}.$$

Checking the properties, observe that we have a (local) maximum at $\tau = 0$, and that the function is symmetric.

The Homogeneous Poisson Counting Process

Let $T = [0,\infty)$. Suppose events occur randomly in time in the following fashion:

1. The number of events occurring in non-overlapping intervals of time are independent.
2. The probability of one event exactly in any interval of length Δt is equal to $\lambda\Delta t + o(\Delta t)$ for Δt sufficiently small.

   $$\frac{o(\Delta t)}{\Delta t} \to 0 \text{ as } \Delta t \to 0.$$

   (That is, $o(\Delta t)$ is the generic term for terms of order higher than Δt.) Also, the probability of more than one event occurring during an interval of length $\Delta t = o(\Delta t)$.

Now define a r.p. $\{X_t, t \in T\}$ by X_t as the number of events occurring in the interval $[0,t]$. Then X_t has the following properties:

1. $X_t - X_s$ is Poisson with parameter $\lambda(t-s)$:

   $$P(X_t - X_s = k) = \frac{(\lambda(t-s))^k e^{-\lambda(t-s)}}{k!}.$$

2. $(X_{t_1} - X_{s_1})$ and $(X_{t_2} - X_{s_2})$ are independent r.v.s for all nonoverlapping intervals $[s_1,t_1]$ and $[s_2,t_2]$.

The parameter λ is called the **rate** of X_t. Property 2 follows from the first assumption. We say that such a process has **independent increments**.

Such a process is called a Poisson counting process (PCP) with rate λ. These two properties complete determine a PCP. All finite-dimensional distributions (fdds) of the process can be determined from these two properties.

How do we show the Poisson distribution property? Pick $t > s \geq 0$. Let

$$p_k(t,s) = \Pr(\text{exactly } k \text{ occurrences in } [s,t])$$

for $k \geq 0$. Then

$$\begin{aligned} p_k(t+\Delta t, s) &= \Pr(k \text{ occurrences in } [s, t+\Delta t]) \\ &= \Pr(k \text{ occurrences in } [s,t]) \Pr(0 \text{ in } [t, t+\Delta t]) + \\ &\quad \Pr(k-1 \text{ occurrences in } [s,t]) \Pr(1 \text{ occurrence in } [t, t+\Delta t]) + \\ &\quad \Pr(\text{fewer than } k-1 \text{ occurrences in } [s,t]) \Pr(\text{all the rest}) \end{aligned}$$

By assumption 2,

$$\begin{aligned} p_k(t+\Delta t, s) &= p_k(t,s)(1-\lambda\Delta t + o(\Delta t)) + p_{k-1}(t,s)(\lambda \Delta t + o(\Delta t)) + (?)(o(\Delta t)) \\ &= \Delta t \lambda (p_{k-1}(t,s) - p_k(t,s)) + p_k(t,s) + o(\Delta t). \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial}{\partial t} p_k(t,s) &= \lim_{\Delta t \to 0} \frac{p_k(t+\Delta t, s) - p_k(t,s)}{\Delta t} \\ &= \lambda[p_{k-1}(t,s) - p_k(t,s)] + \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t}. \end{aligned}$$

So

$$\frac{\partial}{\partial t} p_k(t,s) = \lambda[p_{k-1}(t,s) - p_k(t,s)] \qquad t \geq s.$$

Now $p_{-1}(t,s) = 0$. When $k = 0$ we get

$$\frac{\partial}{\partial t} p_0(t,s) = -\lambda p_0(t,s). \tag{*}$$

so

$$p_0(t,s) = C(s)e^{\lambda t}$$

We have another boundary condition: $p_0(s,s) = 1$, giving

$$p_0(t,s) = e^{-\lambda(t-s)}$$

Now we could proceed solve the set of equations for $k = 1, 2, \ldots$. For example, when $k = 1$:

$$\frac{\partial}{\partial t} p_1(t,s) = \lambda(p_0(t,s) - p_1(t,s))$$

This could be solved, e.g., using Laplace transforms. In general we would find

$$p_k(t,s) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!} \qquad k = 0, 1, \ldots, \qquad t > s \geq 0.$$

As stated, the properties allow us to find all finite dimensional distributions. For example, suppose we want to find the joint distribution of X_{t_1} and X_{t_2} for $t_1 < t_2$.

$$\begin{aligned} P(X_{t_1} = i, X_{t_2} = j) &= P(X_{t_1} = i, X_{t_2} - X_{t_1} = j - i) \\ &= P(X_{t_1} = i) P(X_{t_2} - X_{t_1} = j - i) \\ &= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \frac{(\lambda(t_2 - t_1))^{j-i} e^{-\lambda(t_2 - t_1)}}{(j-i)!} \end{aligned}$$

where the factorization occurs because of independent increments.

Draw a typical sample path...

The process is called *homogeneous* because the rate at which the events occur does not depend on t.

Let us work out the mean and autocorrelation functions.

$$\mu_X(t) = E[X_t] = E[X_t - X_0] = \lambda t$$

(Poisson).

Assume $t > s$:

$$\begin{aligned} E[X_t X_s] &= E[(X_t - X_s)X_s] + E[X_s^2] = E[X_t - X_s]E[X_s] + E[X_s^2] \\ &= [\lambda(t-s)][\lambda s] + [\lambda s + (\lambda s)^2] \\ &= \lambda^2 ts + \lambda s \end{aligned}$$

and if $t < s$, $\lambda^2 ts + \lambda t$.

This process is not WSS! The mean is not constant, and the autocorrelation is not a function of the time difference.

Now create a function $Z_t = X_{t+\Delta t} - X_t$, for some fixed Δt. The random process $\{Z_t\}$ *is* WSS. The increase in the number of counts (over some fixed interval) does not depend on the time.

We could create an *inhomogeneous* Poisson if the probability of an occurrence in the interval $[t, t+\Delta t]$ is $\lambda_t \Delta t + o(\Delta t)$. Then

$$X_t - X_2 \sim \text{Poisson with rate } \int_0^t \lambda_x dx.$$

Gaussian Random Processes

Definition 13 The **Gaussian random process** is a r.p. all of whose finite dimensional distributions are Gaussian. That is, if $\{X_t, t \in T\}$ is a Gaussian r.p., then for all $n \in \mathbb{Z}^+$ and all sample instants $t_1, t_2, \ldots, t_n$, the random vector

$$\begin{bmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{bmatrix}$$

is a multidimensional Gaussian distribution. $\square$

For G.R.P. the entire distribution is completely determined by the mean

$$\begin{bmatrix} \mu_X(t_1) \\ \vdots \\ \mu_X(t_n) \end{bmatrix}$$

and the covariance

$$\begin{bmatrix} \operatorname{cov}(X_{t_1}, X_{t_1}) & \operatorname{cov}(X_{t_1}, X_{t_2}) & \cdots & \operatorname{cov}(X_{t_1}, X_{t_n}) \\ \vdots & & & \\ \operatorname{cov}(X_{t_n}, X_{t_1}) & \operatorname{cov}(X_{t_n}, X_{t_2}) & \cdots & \operatorname{cov}(X_{t_n}, X_{t_n}) \end{bmatrix}$$

which are determined by $\mu_X(t)$ and $R_X(t,s)$. That is, the entire distribution is determined by just the first two moments. It follows, therefore, that a WSS Gaussian process is also strictly stationary.

More properties and definitions

Recall that a second-order process is WSS if its mean function is constant and its autocorrelation function depends only on the difference of its arguments.

Definition 14 Suppose $T = \mathbb{R}$. The **power spectral density** (or **spectrum**) of a WSS process $\{X_t, t \in T\}$ is defined as

$$S_X = \mathcal{F}(R_X)$$

That is, the PSD is the Fourier transform of the autocorrelation function:

$$\boxed{S_X(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R_X(\tau)\, d\tau,}$$

assuming the transform exists.

Suppose $T = \mathbb{Z}$. The power spectral density (or spectrum) is

$$S_X = \mathcal{F}(R_X)$$

where the discrete-time Fourier transform is used,

$$\boxed{S_X(\omega) = \sum_{k=-\infty}^{\infty} e^{-i\omega k} R_X(k),}$$

assuming the transform exists. □

A sufficient condition for the existence of S_X is that $\int_{-\infty}^{\infty} |R_X(\tau)| d\tau < \infty$ or $\sum_{k=-\infty}^{\infty} |R_X(k)| < \infty$.

Example 2 Suppose $R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$. Then

$$S_X(\omega) = \frac{2\beta\sigma^2}{\beta^2 + \omega^2}$$

Such a process is called a **wide-sense Markov** process. Comment on changes with β.

A G.R.P. with this spectrum is called an **Orstein-Uhlenbeck** process. It is sometimes used as a model for wide-band (e.g., nearly white) noise which has been lowpass filtered. □

Example 3 The Ideal Low Pass Process. Suppose

$$S_X(\omega) = \begin{cases} S_0 & |\omega| < \omega_0 \\ 0 & \text{otherwise.} \end{cases}$$

This is a model for wideband noise in the passband of a system of interest.

$$R_X(\tau) = \frac{S_0\omega_0}{\pi} \frac{\sin(\omega_0\tau)}{\omega_0\tau}.$$

Observe (from the sinc function) that there are delays where the signals are uncorrelated, $\pi/\omega_0, 2\pi/\omega_0$, etc. If the signal were Gaussian, it would also be independent. □

Example 4 Let $R_X(k) = \sigma^2 r^{|k|}$ for $|r| < 1$ and $k \in \mathbb{Z}$. Then

$$S_X(\omega) = \frac{\sigma^2(1 - r^2)}{1 - 2r\cos\omega + r^2}.$$

□

Properties of Spectra

1. Symmetry:

$$S_X(\omega) = \begin{cases} \int_{-\infty}^{\infty} \cos(\omega\tau) R_X(\tau)\, d\tau & T = \mathbb{R} \\ \sum_{k=-\infty}^{\infty} \cos(k\omega) R_X(k) & T = \mathbb{Z}. \end{cases}$$

2. $S_X(\omega) = S_X(-\omega)$. $S_X(\omega) = S_X^*(\omega)$.
3. Inverse:
$$\begin{aligned} R_X(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} S_X(\omega)\,d\omega \qquad\qquad T = \mathbb{R}. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\omega\tau) S_X(\omega)\,d\omega. \end{aligned}$$
$$\begin{aligned} R_X(\tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega\tau} S_X(\omega)\,d\omega \qquad T = \mathbb{Z}. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\omega\tau) S_X(\omega)\,d\omega. \end{aligned}$$
4. If $T = \mathbb{R}$,
$$R_X(0) = E[X_t^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\,d\omega.$$
if $T = \mathbb{Z}$:,
$$R_X(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\omega)\,d\omega.$$
5. $S_X(\omega) \geq 0$ for all ω. (This follows from the non-negative definiteness of R_X.

   Any symmetric non-negative definite function having a finite integral is a legitimate spectral density.

   Observe that being nnd and finite integral is analogous to a probability density, so that makes R_X analogous to a characteristic function.

Cases when $R_X(\tau)$ does not have a transform

Example 5 Recall the random sinusoid had
$$R_X(\tau) = \frac{E[A^2]}{2}\cos(\omega_0\tau).$$
This is a periodic function. As a result, it can be described using a Fourier series. However, if we restrict ourselves to conventional functions (as opposed to δ functions), there is no Fourier transform. □

We examine this and other cases that are WSS but do not have a Fourier transform (in the conventional sense).

1. Suppose $T = \mathbb{R}$. Then $R_X(\tau)$ is continuous and is the autocorrelation function of a WSS r.p. if and only if there is a c.d.f. G_X satisfying $G_X(b) = 1 - G_X(b)$ such that
$$R_X(\tau)/R_X(0) = \int_{-\infty}^{\infty} e^{i\omega\tau} dG_X(\omega).$$
   This transform is called the **Fourier-Stieltjes** transform.
2. Suppose $T = \mathbb{Z}$. Then $R_X(k)$ is the autocorrelation function of a WSS r.p. if and only if there exists a G_X satisfying $G_X(b) = 1 - G_X(b)$ such that
$$R_X(k)/R_K(0) = \int_{-\pi}^{\pi} e^{i\omega k} dG_X(\omega)$$

Thus the a.c.f. acts like a characteristic function, but also has symmetry.

If $S_X(\omega)$ exists, then

$$G_X(\omega) = \int_{-\infty}^{\infty} S_X(\zeta) d\zeta / (2\pi R_X(0)).$$

In this case, $S_X(\omega)/2\pi R_X(0)$ is, in fact, a p.d.f.

More generally, $2\pi R_X(0) G_X$ is a spectral distribution of $\{X_t\}$.

Example 6 Random sinusoid.

$$R_X(\tau)/R_X(0) = \cos(\omega_0 \tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} dG_X(\omega).$$

where

$$G_X(\omega) = \frac{1}{2}[u(\omega + \omega_0) + u(\omega - \omega_0)]$$

(right-continuous). □

Joint properties of Two Random Processes

Suppose we have two r.p.s $\{X_t, t \in T\}$, $\{Y_t, t \in T\}$.

Definition 15 The **cross correlation** function is

$$R_{XY}(t, s) = E[X_t Y_s].$$

□

Properties of the cross correlation function:

1. $R_{XY}(t, s) = R_{YX}(s, t)$ (symmetry)
2. $|R_{XY}(t, s)| \leq \sqrt{R_X(t, t) R_Y(s, s)}$ (Schwartz inequality)
3. $|R_{XY}(t, s)| \leq \frac{1}{2}[R_X(t, t) + R_Y(s, s)]$

Definition 16 The random processes X_t and Y_t are **orthogonal** if $R_{XY}(t, s) = 0$ for all $s, t \in T$. □

Definition 17 The **cross-covariance** function of X_t and Y_t is

$$C_{XY}(t, s) = \text{cov}(X_t, Y_s).$$

□

Definition 18 The random processes X_t and Y_t are **jointly wide-sense stationary** if $\mu_X(t)$ and $\mu_Y(t)$ are constant and $R_{XY}(t+h, t)$ is independent of t for all $t \in T$ and for all $h \in T$. In this case, we write $R_{XY}(t + h, t) = R_{XY}(h)$. □

Properties of $R_{XY}(h)$ for jointly WSS:

1. $R_{XY}(0) = R_{YX}(0)$
2. $R_{XY}(h) = R_{YX}(-h)$
3. If X and Y are individually WSS, then $|R_{XY}(h)| \leq \sqrt{R_X(0) R_Y(0)}$. and $|R_{XY}(h)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$.

Definition 19 If X_t and Y_t are jointly WSS random processes, the **cross power spectrum** is defined as

$$S_{XY}(\omega) = \mathcal{F}[R_{XY}(\tau)] = \begin{cases} \int_{-\infty}^{\infty} e^{-i\omega\tau} R_{XY}(\tau)\, d\tau & T = \mathbb{R} \\ \sum_{k=-\infty}^{\infty} e^{-i\omega k} R_{XY}(k) & T = \mathbb{Z}. \end{cases}$$

□

Properties of Spectra:

1. $S_{XY}(\omega) = S_{XY}^*(\omega)$.
2. If X_t and Y_t are individually and jointly WSS then
$$S_{XY}(\omega)|^2 \leq S_X(\omega) S_Y(\omega).$$
3. $\operatorname{Re} S_{XY}(\omega) = \operatorname{Re} S_{XY}(-\omega)$ and $\operatorname{Im}(S_{XY}(\omega)) = -\operatorname{Im}(S_{XY}(\omega))$.

Uncorrelated and independent

Definition 20 Two random processes are **uncorrelated** if $C_{XY}(t,s) = 0$ for all $s, t \in T$. □

Definition 21 The random processes X_t and Y_t are **independent** if $(X_{t_1}, \ldots, X_{t_n})$ and $(Y_{s_1}, \ldots, Y_{s_m})$ are independent random vectors for all

$$n, m \in \mathbb{Z}^+$$

and all $t_1, \ldots, t_n, s_1, \ldots, s_m \in T$. □

Note: Independence implies uncorrelated. The converse is not true.

Definition 22 The random processes X_t and Y_t are **jointly Gaussian** if $(X_{t_1}, \ldots, X_{t_n}, Y_{s_1}, \ldots, Y_{s_m})$ is a Gaussian random vector for all $n, m \in \mathbb{Z}^+$ and all $t_1, \ldots, t_n, s_1, \ldots, s_m \in T$. □

For jointly Gaussian random processes, we can characterize by a mean vector and a covariance matrix. All f.d.d.s are determined by $\mu_x(t)$, $\mu_y(t)$, $R_X(t,s)$, $R_Y(t,s)$ and $R_{XY}(t,s)$.

For this case, it is true that uncorrelated implies independence.