Utah State University
ECE 6010
Stochastic Processes
Homework # 9 Solutions

1. **Suppose $\{X_t, t \in \mathbb{R}\}$ is a ramdom process with power spectral density**

$$S_X(\omega) = \frac{1}{(1+\omega^2)^2}.$$

Find the autocorrelation function of X_t.

$$\begin{aligned}
R_X(\tau) &= \mathcal{F}^{-1}\{S_X(\omega)\} = \mathcal{F}^{-1}\left\{\frac{1}{(1+\omega^2)}\right\} * \mathcal{F}^{-1}\left\{\frac{1}{(1+\omega^2)}\right\} \\
&= \frac{1}{2}e^{-|\tau|} * \frac{1}{2}e^{-|\tau|} \\
&= \frac{1}{4}\int_{-\infty}^{\infty} e^{-|t|}e^{-|\tau-t|}\,dt \\
&= \frac{1}{4}\int_{-\infty}^{0} e^{t}e^{t-\tau}\,dt + \frac{1}{4}\int_{0}^{\tau} e^{-t}e^{t-\tau}\,dt + \frac{1}{4}\int_{\tau}^{\infty} e^{-t}e^{\tau-t}\,dt \qquad (\text{for } \tau \geq 0) \\
&= \frac{1}{4}\left[\int_{-\infty}^{0} e^{2t-\tau}\,dt + \int_{0}^{\tau} e^{-\tau}\,dt + \int_{\tau}^{\infty} e^{t-2\tau}\,dt\right] \\
&= \frac{1}{4}\left[\frac{1}{2}e^{-\tau} + \tau e^{-\tau} + \frac{1}{2}e^{-\tau}\right] \\
&= \frac{1}{4}\left(\tau e^{-\tau} + e^{-\tau}\right) \qquad (\text{for } \tau \geq 0)
\end{aligned}$$

Similarly,

$$R_X(\tau) = \frac{1}{4}\left(-\tau e^{\tau} + e^{\tau}\right) \qquad (\text{for } \tau < 0)$$

Therefore,

$$R_X(\tau) = \frac{1}{4}e^{-|\tau|}(|\tau| + 1)$$

2. **Suppose that ω is a random variable with p.d.f. f_ω and θ is a random variable independent of ω uniformly distributed in $(-\pi, \pi)$. Define a random process by $X_t = a\cos(\omega t + \theta),\ t \in \mathbb{R}$ where a is a constant. Find the power spectral density of $\{X_t\}$.**

$$\begin{aligned}
E[X_{t_1}X_{t_2}] &= E\{a^2\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\} \\
&= \frac{1}{2}a^2 E\{\cos(\omega t_1 - \omega t_2) - \cos(\omega t_1 + \omega t_2 + 2\theta)\} \\
&= \frac{1}{2}a^2\left[E\{\cos(\omega t_1 - \omega t_2)\} - \underbrace{E\{\cos(\omega t_1 + \omega t_2 + 2\theta)\}}_{0}\right] \\
&= \frac{1}{2}a^2 E\{\cos(\omega t_1 - \omega t_2)\} \\
&= \frac{1}{2}a^2\int_{-\infty}^{\infty}\cos(\tau\omega)f_\omega(\omega)\,d\omega = \frac{1}{2}a^2\int_{-\infty}^{\infty}\frac{e^{j\omega\tau} + e^{-j\omega\tau}}{2}f_\omega(\omega)\,d\omega \\
&= \frac{1}{4}a^2\ 2\pi[\mathcal{F}^{-1}\{f_\omega(\omega)\} + \mathcal{F}^{-1}\{f_\omega(-\omega)\}] \\
&= \frac{\pi a^2}{2}[\mathcal{F}^{-1}\{f_\omega(\omega)\} + \mathcal{F}^{-1}\{f_\omega(-\omega)\}]
\end{aligned}$$

Therefore,

$$S_X(\omega) = \frac{\pi a^2}{2}[f_\omega(\omega) + f_\omega(-\omega)]$$

3. **Suppose events occur randomly in $T = [0, \infty)$ in the following way:**

   (a) **The numbers of events in nonoverlapping intervals are independent of one another.**

   (b) P(**exactly one event in** $(t, t + \Delta t)) = \lambda(t)\Delta t + o(\Delta t)$**, where** $\lambda(t)$ **is a continuous nonnegative function on** $[0, \infty)$**.**

   (c) P(**more than one event in an interval of length** $\Delta t) = o(\Delta t)$.

   Define a random process $\{X_t, t \in T\}$ **by** $X_0 = 0$ **and** x_t **i s the number of events occurring in** $(0, t]$**.**

   (a) **For** $t > s \geq 0$**, show that** $(X_t - X_s)$ **is a Poisson random variable with parameter** $\int_z^t \lambda(x)\ dx$**.**

   $P(\text{k events in } [s, t]) = p_k(t, s)$ for $t > s \geq 0$ and $k \geq 0$.

$$\begin{aligned} p_k(t + \Delta t, s) &= p_k(t, s)(1 - \lambda(t)\Delta t + o(\Delta t)) + p_{k-1}(t, s)(\lambda(t)\Delta t + o(\Delta t)) + p_{<k-1}(t, s)o(\Delta t) \\ &= p_k(t, s)(1 - \lambda(t)\Delta t + o(\Delta t)) + p_{k-1}(t, s)(\lambda(t)\Delta t + o(\Delta t)) + o(\Delta t) \end{aligned}$$

$$\frac{d}{dt}p_k(t, s) = \lim_{\Delta t \to 0} \frac{p_k(t + \Delta t, s) - p_k(t, s)}{\Delta t} = -p_k(t, s)\lambda(t) + p_{k-1}(t, s)\lambda(t)$$

   If $p_k(t, s) = \frac{e^{-\int_s^t \lambda(q)dq}\left(\int_s^t \lambda(q)dq\right)^k}{k!}$ satisfies the above equation then we can say that $X_t - X_s$ is a Poisson random variable with parameter $\int_s^t \lambda(q)\ dq$.
   Now, let $I = \int_s^t \lambda(q)\ dq$ and $I' = \lambda(t)$
   Therefore,

$$\begin{aligned} \frac{d}{dt}p_k(t, s) &= \frac{-I'e^I I^k + e^I k I^{k-1} I'}{k!} \\ &= -\frac{I'e^I I^k}{k!} + \frac{e^I I^{k-1} I'}{(k-1)!} \\ &= \frac{e^I I^k}{k!}\lambda(t) + \frac{e^I I^{k-1}}{(k-1)!}\lambda(t) \\ &= -p_k(t, s)\lambda(t) + p_{k-1}(t, s)\lambda(t) \end{aligned}$$

   So that $p_k(t, s)$ does satisfy. And $X_t - X_s$ is Poisson.

   (b) **Find the mean and autocorrelation functions of** $\{X_t\}$**.**

   We have, $E[X_t - X_s] = \sum_{k=0}^{\infty} k p_k(t, s)$. So,

$$\begin{aligned} E[X_t] &= E[X_t - X_0] = \sum_{k=0}^{\infty} k p_k(t, 0) \\ &= \sum_{k=0}^{\infty} k \frac{e^{-\int_0^t \lambda(q)dq}\left(\int_0^t \lambda(q)dq\right)^k}{k!} \\ &= \int_0^t \lambda(q)\ dq \end{aligned}$$

Now,

$$\begin{aligned}
E[X_tX_s] &= E[X_t(X_t+X_s-X_t)] = E[X_t^2]+E[X_t(X_s-X_t)] \\
&= E[X_t^2]+E[X_tY_t] = E[X_t^2]+E[X_t]E[Y_t] \\
&= \lambda_X^2+\lambda_X+\lambda_X\lambda_Y = \lambda_X(\lambda_X+1+\lambda_Y) \\
&= \int_0^t \lambda(q)dq\left[\int_0^t \lambda(q)dq+1+\int_t^s \lambda(q)dq\right]
\end{aligned}$$

So,

$$R_X(t,s) = \int_0^t \lambda(q)dq\left[\int_0^s \lambda(q)\ dq+1\right] \qquad (\text{for } s>t)$$

and

$$R_X(t,s) = \int_0^s \lambda(q)dq\left[\int_0^t \lambda(q)\ dq+1\right] \qquad (\text{for } t>s)$$

4. **Suppose that $\{X_t, t\in\mathbb{R}\}$ is a w.s.s., zero-mean, Gaussian random process with auto-correlation function $R_X(\tau), \tau\in\mathbb{R}$ and power spectral density $S_X(\omega), \omega\in\mathbb{R}$. Define the random process $\{Y_t, t\in\mathbb{R}\}$ by $Y_t=(X_t)^2, t\in\mathbb{R}$. find the mean, autocorrelation, and powerspectral density of $\{Y_t, t\in\mathbb{R}\}$.**

   Mean :

$$\mu_y(t) = E[X_t^2] = R_X(0) = \sigma^2$$

   Autocorrelation :

$$\begin{aligned}
R_Y(t,s) &= E[Y_tY_s] = E[X_t^2X_s^2] \\
&= \frac{\partial^4}{\partial u^2\partial v^2}\Phi_{X_tX_s}(u,v)\Big|_{u=v=0}\frac{1}{i^4} \\
&\ \ \vdots \\
&= R_X^2(0)+2R_X^2(\tau) \qquad (\tau=t-s)
\end{aligned}$$

   PSD :

$$S_Y(\omega) = \mathcal{F}\{R_Y(\tau)\} = 2\pi R_X^2(0)\delta(\omega)-2(S_X(\omega)*S_X(\omega))$$

5. **Suppose U and V are independent random variables with $E[U]=E[V]=0$ and $\text{var}(U)=\text{var}(V)=1$. Define random processes by**

$$X_t = U\cos t+V\sin t \quad Y_t = U\sin t+V\cos t, \quad t\in\mathbb{R}.$$

   Find the autocorrelation and cross-correlation functions of $\{X_t, t\in\mathbb{R}\}$ and $\{Y_t, t\in\mathbb{R}\}$. Are $\{X_t\}$ and $\{Y_t\}$ jointly wide sense stationary? Are they individually wide sense stationary?

$$\begin{aligned}
R_X(t,s) &= E[X_tX_s] = E[(U\cos t+V\sin t)(U\cos s+V\sin s)] \\
&= E[U^2\cos t\cos s+UV(\cos t\sin s+\sin t\cos s)+V^2\sin t\sin s] \\
&= \cos t\cos sE[U^2]+E[U]E[V](\cos t\sin s+\sin t\cos s)+E[V^2]\sin t\sin s \\
&= \cos t\cos s+\sin t\sin s \\
&= \cos(t-s)
\end{aligned}$$

   Similarly,

$$R_Y(t,s) = cos(t-s)$$

$$
\begin{aligned}
R_{XY}(t,s) &= E[X_tY_s] = E[(U\cos t + V\sin t)(U\sin s + V\cos s)] \\
&= E[U^2\cos t\sin s + UV(\cos t\cos s + \sin t\sin s) + V^2\sin t\cos s] \\
&= E[U^2]\cos t\sin s + E[UV](\cos t\cos s + \sin t\sin s)E[V^2]\sin t\cos s \\
&= \cos t\sin s + \sin t\cos s \\
&= \sin(t+s)
\end{aligned}
$$

$$
\mu_X(t) = 0 \qquad \mu_Y(t)
$$

So, $\{X_t\}$ and $\{Y_t\}$ are individually WSS, but not jointly WSS.