Utah State University
ECE 6010
Stochastic Processes
Homework # 8 Solutions

1. Suppose $\{X_t, t \geq 0\}$ is a Wiener process. Define a process $\{Y_t, t \geq 0\}$ by $Y_t = X_{t+D} - X_t$ for a fixed positive number D.

   (a) Find the mean and autocorrelation functions of $\{Y_t\}$.
   Mean :

$$\mu_Y(t) = E[Y_t] = E[X_{t+D}] - E[X_t] = \mu(t+D) - \mu(t) = \mu D$$

   Autocorrelation :

$$\begin{aligned}
R_Y(t,s) &= E[(X_{t+D} - X_t)(X_{s+D} - X_s)] \\
&= E[X_{t+D}X_{s+D}] - E[X_{t+D}X_s] - E[X_tX_{s+D}] + E[X_tX_s] \\
&= [\sigma^2 \min(t+D, s+D) + \mu^2(t+D)(s+D)] - [\sigma^2 \min(t+D, s) + \mu^2(t+D)s] \\
&\quad -[\sigma^2 \min(t, s+D) + \mu^2(S+D)t] + [\sigma^2 \min(t,s) + \mu^2 st] \\
&= \sigma^2[\min(t+D, s+D) - \min(t+D, s) - \min(t, s+D) + \min(t,s)] \\
&\quad +\mu^2[ts + Dt + sD + D^2 - st - sD - st - Dt + st] \\
&= \begin{cases} \sigma^2[s + D - s - \min(t, s+D) + s] + \mu D^2 & t \geq s \\ \sigma^2[t + D - t - \min(t, s+D) + t] + \mu D^2 & t < s \end{cases} \\
&= \begin{cases} \mu^2 D^2 & t \geq s\ ,\ t - s - D \geq 0 \\ \sigma^2[s - t + D] + \mu^2 D^2 & t \geq s\ ,\ t - s - D < 0 \\ \mu^2 D^2 & t < s\ ,\ t - s + D < 0 \\ \sigma^2[t - s + D] + \mu^2 D^2 & t < s\ ,\ t - s + D \geq 0 \end{cases} \\
&= \begin{cases} \mu^2 D^2 & |t-s| \geq D \\ \sigma^2[D - |t-s|] + \mu^2 D^2 & |t-s| < D \end{cases}
\end{aligned}$$

   So $R_Y(t,s) = \sigma^2\, ax(0, D - |t-s|) + \mu^2 D^2$.

   (b) Show that $\{Y_t\}$ is a stationary and find its spectrum.
   $\mu_y(t)$ is a constant and $R_Y(t,s) = R_Y(t-s)$ $\Rightarrow$ W.S.S.
   Since Gaussian too $\Rightarrow$ Strictly Stationary.

$$R_Y(\tau) = \mu^2 D^2 + \left[\text{rect}\left(\frac{2\tau}{D}\right) * \text{rect}\left(\frac{2\tau}{D}\right)\right]\sigma^2 D$$

   So,

$$S_Y(\omega) = \mathcal{F}\{R_Y(\tau)\} = 2\pi\mu^2 D^2 \delta(\omega) + \sigma^2 D \frac{\sin^2(\omega D/2)}{(\omega D/2)^2}$$

2. Suppose $\{X_t\}$ and $\{Y_t\}$ are zero mean and individually and jointly W.S.S. Show that the mean-square error associated with the noncausal Wiener filter for estimation of X_t from $\{Y_t, t \in \mathbb{R}\}$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_X(\omega) - \frac{|S_{XY}(\omega)|^2}{S_Y(\omega)}\right]d\omega.$$

$$\begin{aligned}
MSE &= E[(X_t - \hat{X}_t)^2] \\
&= E[(X_t - \hat{X}_t)X_t] - \underbrace{E[(X_t - \hat{X}_t)\hat{X}_t]}_{0} \\
&= E[X_t^2] - E[X_t\hat{X}_t] \\
&\qquad (E[(X_t - \hat{X}_t)\hat{X}_t] = 0 \ \Rightarrow \ E[X_t\hat{X}_t] = E[\hat{X}_t^2]) \\
&= E[X_t^2] - E[\hat{X}_t^2] \\
&= R_X(0) - R_{\hat{X}}(0) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (S_X(\omega) - S_{\hat{X}}(\omega)) \ d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (S_X(\omega) - |H_0(\omega)|^2 S_Y(\omega)) \ d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (S_X(\omega) - \frac{|S_{XY}(\omega)|^2}{|S_Y(\omega)|^2} S_Y(\omega)) \ d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (S_X(\omega) - \frac{|S_{XY}(\omega)|^2}{S_Y(\omega)}) \ d\omega
\end{aligned}$$

3. Suppose $Y_t = S_t + N_t$ for $t \in \mathbb{R}$, where $\{S_t\}$ and $\{N_t\}$ are zero-mean, W.S.S., and orthogonal. Suppose that we wish to estimate $X_t = \int_{-\infty}^{\infty} k(t-\tau)S_\tau \ d\tau$, with an estimate of the form $\hat{X}_t = \int_{-\infty}^{\infty} h(t-\tau)Y_\tau \ d\tau$, where k and h are impulse responses of linear time-invariant systems. show that
$$E[(X_t - \hat{X}_t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[|K(\omega) - H(\omega)|^2 S_S(\omega) + |H(\omega)|^2 S_N(\omega)\right] \ d\omega$$
where K and H are the transfer functions of k and h, respectively, and S_S and S_N are the power spectral densities of $\{S_t\}$ and $\{N_t\}$. (Note the case that $k(t) = \delta(t - \lambda)$ for some fixed $\lambda \in \mathbb{R}$.)

$$\begin{aligned}
E[(X_t - \hat{X}_t)^2] &= E\left[\left(\int_{-\infty}^{\infty} k(t-\tau)S_\tau \ d\tau - \int_{-\infty}^{\infty} h(t-\tau)Y_\tau \ d\tau\right)^2\right] \\
&= E\left[\left(\int_{-\infty}^{\infty} k(t-\tau)S_\tau \ d\tau - \int_{-\infty}^{\infty} h(t-\tau)S_\tau \ d\tau - \int_{-\infty}^{\infty} h(t-\tau)N_\tau \ d\tau\right)^2\right] \\
&= E\left[\left(\int_{-\infty}^{\infty} (k(t-\tau) - h(t-\tau))S_\tau \ d\tau - \int_{-\infty}^{\infty} h(t-\tau)N_\tau \ d\tau\right)^2\right] \\
&= E\left[\left(\int_{-\infty}^{\infty} (k(t-\tau) - h(t-\tau))S_\tau \ d\tau\right)^2\right] + E\left[\left(\int_{-\infty}^{\infty} h(t-\tau)N_\tau \ d\tau\right)^2\right] \\
&= R_W(0) + R_Z(0) \\
&\qquad \begin{pmatrix} \text{Where} & W(t) = \int_{-\infty}^{\infty} (k(t-\tau) - h(t-\tau))S_\tau \ d\tau \\ \text{and} & Z(t) = \int_{-\infty}^{\infty} h(t-\tau)N_\tau \ d\tau \end{pmatrix} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_W(\omega) \ d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Z(\omega) \ d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[|K(\omega) - H(\omega)|^2 S_S(\omega) + |H(\omega)|^2 S_N(\omega)\right] \ d\omega
\end{aligned}$$

4. Consider the situation of the previous problem with $k(t) = \delta(t - \lambda)$,

$$S_S(\omega) = \frac{A^2}{\alpha^2 + \omega^2} \qquad S_N(\omega) = \frac{N_0}{2}$$

(a) Find the noncausal Wiener filter for estimating X_t from $\{Y_t, t \in \mathbb{R}\}$. Find the corresponding mean-square error.

When $k(t) = \delta(t - \lambda)$,

$$X_t = \int_{-\infty}^{\infty} \delta(t - \lambda - \tau) S_\tau \; d\tau = S_{t-\lambda}$$

and

$$K(\omega) = \mathcal{F}\{\delta(t - \lambda)\} = e^{-j\lambda\omega}$$

$$\begin{aligned}
R_Y(t,u) &= E[(S_t + N_t)(S_u + N_u)] \\
&= E[S_t S_u + S_t N_u + N_t S_u + N_t N_u] \\
&= R_S(\tau) + R_{SN}(\tau) + R_{SN}(-\tau) + R_N(\tau) \\
S_Y(\omega) &= S_S(\omega) + S_N(\omega) + S_{SN}(\omega) + S_{SN}(-\omega) \\
&= S_S(\omega) + S_N(\omega) + 2 \ \mathrm{Re}[S_{SN}(\omega)] \\
&\qquad \text{(Since } S \perp N,\ S_{SN} = 0\text{)} \\
&= S_S(\omega) + S_N(\omega)
\end{aligned}$$

$$\begin{aligned}
R_{XY}(t,u) &= E[X_t Y_u] \\
&= E[S_{t-\lambda} S_u] + E[S_{t-\lambda} N_u] \\
&= R_s(t - \lambda - u) + R_{SN}(t - \lambda - u) \\
R_{XY}(\tau) &= R_s(\tau - u) + R_{SN}(\tau - u) \\
S_{XY}(\omega) &= e^{-j\omega\lambda} S_S(\omega) + e^{-j\omega\lambda} S_{SN}(\omega) \\
&\qquad \text{(Since } S \perp N,\ R_{SN} = S_{SN} = 0\text{)} \\
&= e^{-j\omega\lambda} S_S(\omega)
\end{aligned}$$

$$\begin{aligned}
H_0(\omega) &= \frac{S_{XY}(\omega)}{S_Y(\omega)} \\
&= \frac{e^{-j\omega\lambda} S_S(\omega)}{S_S(\omega) + S_N(\omega)} \\
&= \frac{2A^2}{N_0} \frac{1}{\omega^2 + \beta^2} e^{-j\omega\lambda} \qquad \text{(where } \beta^2 = \alpha^2 + \tfrac{2A^2}{N_0}\text{)}
\end{aligned}$$

$$h_0(t) = \mathcal{F}^{-1}\{H_0(\omega)\} = \frac{A^2}{\beta N_0} e^{-\beta|t-\lambda|}$$

From the previous problem

$$
\begin{aligned}
MSE &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[|K(\omega)-H(\omega)|^2 S_S(\omega)+|H(\omega)|^2 S_N(\omega)\right]\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{-j\omega\lambda}-\frac{e^{-j\omega\lambda}S_S(\omega)}{S_S(\omega)+S_N(\omega)}\right|^2 S_S - \left|\frac{e^{-j\omega\lambda}S_S(\omega)}{S_S(\omega)+S_N(\omega)}\right|^2 S_N\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_N^2(\omega)}{(S_S(\omega)+S_N(\omega))^2}S_S+\frac{S_S^2(\omega)}{(S_S(\omega)+S_N(\omega))^2}S_N\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_N^2(\omega)S_S(\omega)+S_S^2(\omega)S_N(\omega)}{(S_S(\omega)+S_N(\omega))^2}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_S(\omega)S_N(\omega)}{S_S(\omega)+S_N(\omega)}\, d\omega
\end{aligned}
$$

(b) Find the causal Wiener filter for estimating X_t from $\{Y_\tau, \tau \le t\}$. Consider $\lambda < 0$ and $\lambda \ge 0$.
We have

$$H_0(\omega) = \frac{1}{S_Y^+(\omega)}\left[\frac{S_{XY}(\omega)}{S_Y^-(\omega)}\right]_+$$

where

$$S_Y(\omega) = S_S(\omega)+S_N(\omega) = \frac{A^2}{\alpha^2+\omega^2}+\frac{N_0}{2} = \frac{A^2+\alpha^2N_0/2+\omega^2N_0/2}{\alpha^2+\omega^2} = \frac{N_0/2(\omega+i[\alpha^2+2A^2/N_0]^{1/2})(\omega}{(\omega+i\alpha)(\omega -}$$

so

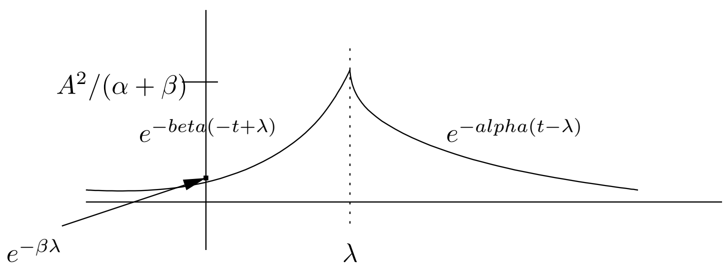
$$S_Y^+(\omega) = \sqrt{N_0/2}\frac{\omega-i[\alpha^2+2A^2/N_0]^{1/2}}{\omega-i\alpha} \qquad S_Y^-(\omega) = \sqrt{N_0/2}\frac{\omega+i[\alpha^2+2A^2/N_0]^{1/2}}{\omega+i\alpha}$$

Now

$$H_2(\omega) \triangleq \frac{S_{XY}(\omega)}{S_Y^-(\omega)} = \frac{A^2}{\alpha^2+\omega^2}\frac{\alpha-i\omega}{\beta-i\omega} = \frac{A^2}{\alpha+\beta}\left[\frac{1}{\alpha+i\omega}+\frac{1}{\beta-i\omega}\right]e^{-i\omega\lambda} \qquad \text{(partial fraction expansio}$$

Taking the inverse Laplace transform,

$$h_2(t) = \frac{A^2}{\alpha+\beta}\left[e^{-\alpha(t-\lambda)}u(t-\lambda)+e^{-\beta(-t+\lambda)}u(-t+\lambda)\right]$$



Let

$$H_2(\omega) = \frac{e^{-i\omega\lambda}S_S(\omega)}{S_S(\omega)+S_N(\omega)}$$

(the thing whose $[\;]_+$ we need to compute. Then

$$H_2(\omega) = e^{-i\omega\lambda}\frac{2A^2}{N_0}\frac{1}{(2A^2/N_0+\alpha^2)+\omega^2} = e^{-i\omega\lambda}\frac{2A^2}{N_0}\frac{1}{\beta^2+\omega^2},$$

where $\beta^2 = 2A^2/N_0 + \alpha^2$. Then

$$h_2(t) = \frac{2A^2}{2N_0\beta} e^{-\beta|t-\lambda|} = \begin{cases} \frac{A^2}{N_0\beta} e^{-\beta t} e^{\beta\lambda} & t \geq \lambda \\ \frac{A^2}{N_0\beta} e^{\beta t} e^{-\beta\lambda} & t < \lambda \end{cases}$$

For $\lambda > 0$ we have delay, so that the filter performs *smoothing*.

$$h_1(t) = [h_t(2)]_+ = e^{-\alpha(t-\lambda)} u(t-\lambda) + e^{-\beta(-t+\lambda)} [u(-t-\lambda) - u(-t)] \frac{A^2}{\alpha+\beta}$$

Transforming

$$H_1(\omega) = H_2(\omega) - \frac{A^2}{\alpha+\beta} e^{-\beta\lambda} \frac{1}{b - i\omega} \qquad \text{(throw away the noncausal part)}$$

Then

$$H_0(\omega) = \frac{1}{\sqrt{N_0/2}} \frac{\omega - i\alpha}{\omega - i[\alpha^2 + 2A^2/N_0]^{1/2}} \left[H_2(\omega) - \frac{A^2}{\alpha+\beta} e^{-\beta\lambda} \frac{1}{b - i\omega} \right]$$