Utah State University
ECE 6010
Stochastic Processes
Homework # 7 Solutions

1. Suppose $\{X_t, t \geq 0\}$ is a homogeneous Poisson process with parameter λ. Define a random variable τ as the time of the first occurrence of an event. Find the p.d.f. and the mean of τ.

   p.d.f. :
   We have,
   $$P(X_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$
   So, $P(\tau > t) = P(X_t = 0) = e^{-\lambda t}$
   and $P(\tau \leq t) = 1 - e^{-\lambda t}$, i.e. $F_\tau(t) = 1 - e^{-\lambda t}$.

   Therefore,
   $$f_\tau(t) = \lambda e^{-\lambda t}$$

   Mean :
   $$E[\tau] = \int_0^\infty \tau\lambda e^{-\lambda\tau}\, d\tau = -\tau e^{-\lambda\tau} - \frac{1}{\lambda}e^{-\lambda\tau}\Big|_0^\infty = -0 - 0 + 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$$

2. Suppose $\{X_t, t \in \mathbb{R}\}$ is a w.s.s. random process with autocorrelation function $R_X(\tau)$. Show that if R_X is continuous at $\tau = 0$ then it is continuous for all $\tau \in \mathbb{R}$.

   $$\begin{aligned}
   [R_X(\tau+\delta) - R_X(\tau)]^2 &= (E[X_{\tau+\delta}X_0] - E[X_\tau X_0])^2 \\
   &= (E[X_0(X_{\tau+\delta} - X_\tau)])^2 \\
   &\leq E[X_0^2]E[(X_{\tau+\delta} - X_\tau)^2] \quad \text{(Cauchy-Schwartz)} \\
   &= R_X(0)[R_X(0) - 2R_X(\delta) + R_X(0)] \\
   &= 2R_X(0)[R_X(0) - R_X(\delta)] \\
   &= \leq 2R_X(0)\varepsilon
   \end{aligned}$$

   So,
   $$\Rightarrow |R_X(\tau+\delta) - R_X(\tau)| \leq \sqrt{2R_X(0)\varepsilon}$$
   $$\Rightarrow |R_X(\tau+\delta) - R_X(\tau)| < \varepsilon'$$
   So continuous at all τ.

3. Under the conditions of problem 2, show that for $a > 0$,
   $$P(|X_{t+\tau} - X_t| \geq a) \leq \frac{2(R_X(0) - R_X(\tau))}{a^2}.$$

   Let $g(x) = x^2$. g is non-negative, non-decreasing on $[0, \infty)$ and symmetric about 0. Then,
   $$P(|X| \geq b) \leq \frac{E[g(x)]}{b^2}.$$

Now, our example corresponds to:

$$\begin{aligned} P(|X_{t+\tau} - X_t| \geq a) &\leq \frac{E[|X_{t+\tau} - X_t|^2]}{a^2} \\ &= \frac{R_X(0) - 2R_X(\tau) + R_X(0)}{a^2} \\ &= \frac{2(R_X(0) - R_X(\tau))}{a^2} \end{aligned}$$

4. Suppose A and B are random variables with $E[A^2] < \infty$ and $E[B^2] < \infty$. Define the random processes $\{X_t, t \in \mathbb{R}\}$ and $\{Y_t, t \in \mathbb{R}\}$ by

$$X_t = A + Bt \qquad Y_t = B + At, \quad t \in \mathbb{R}.$$

Find the mean, autocorrelation, and cross correlations of these random processes in terms of the moments of A and B.

Mean :

$$\mu_X(t) = E[A + Bt] = E[A] + tE[B]$$
$$\mu_Y(t) = E[B + At] = E[B] + tE[A]$$

Autocorrelation :

$$\begin{aligned} R_X(t,s) &= E[X_t X_s] \\ &= E[(A + Bt)(A + Bs)] \\ &= E[A^2 + ABs + ABt + B^2 ts] \\ &= E[A^2] + E[AB](s + t) + E[B^2]ts \end{aligned}$$

Similarly,

$$R_Y(t,s) = E[B^2] + E[AB](s + t) + E[A^2]st$$

Cross correlation :

$$\begin{aligned} R_{XY}(t,s) &= E[X_t Y_s] \\ &= E[(A + Bt)(B + As)] \\ &= E[AB + A^2 s + B^2 t + ABts] \\ &= E[AB](1 + st) + E[A^2]s + E[B^2]t \end{aligned}$$

5. Homogeneous Poisson

Simply take $p_k(t,s)$ and substitute it back into the differential equation and show that it works.

$$\frac{\partial}{\partial t} p_k(t,s) = \frac{\partial}{\partial t} \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!}$$
$$= \frac{-\lambda e^{-\lambda(t-s)}(\lambda(t-s))^k + e^{-\lambda(t-s)} k\lambda(\lambda(t-s))^{k-1}}{k!}$$
$$= \lambda \left[\frac{-e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!} + \frac{e^{-\lambda(t-s)}(\lambda(t-s))^{k-1}}{(k-1)!} \right]$$
$$= \lambda[p_{k-1}(t,s) - p_k(t,s)]$$

6. Inhomogeneous Poisson

Simply take $p_k(t,s)$ and substitute it back into the differential equation and show that it works.

$$\begin{aligned}\frac{\partial}{\partial t}p_k(t,s) &= \frac{-\lambda_t e^{-\int_s^t \lambda_x dx} + \lambda_t k(\int_s^t \lambda_x dx)^{k-1}}{k!} \\ &= \lambda_t\left[\frac{-e^{-\int_s^t \lambda_x dx}}{k!} + \frac{(\int_s^t \lambda_x dx)^{k-1}}{(k-1)!}\right] \\ &= \lambda_t[p_{k-1}(t,s) - p_k(t,s)\end{aligned}$$

Mean: $\int_s^t \lambda_x dx$

Covariance: Assume that $t > s$:

$$\begin{aligned}E[X_t X_s] = E[(X_t - X_s)X_s] + E[X_s^2] &= E[X_t - X_s]E[X_s] + E[X_s^2] \\ &= \left[\int_0^t \lambda_x dx + 1\right]\int_0^s \lambda_x dx\end{aligned}$$

Similarly when $t < s$. Then

$$R_X(t,s) = \left[\int_0^s \lambda_x dx + 1\right]\int_0^t \lambda_x dx$$

7. **Suppose $\{X_t, t \in \mathbb{R}\}$ is a random process with power spectral density**

$$S_X(\omega) = \frac{1}{(1+\omega^2)^2}.$$

Find the autocorrelation function of X_t.

$$\begin{aligned}R_X(\tau) &= \mathcal{F}^{-1}\{S_X(\omega)\} = \mathcal{F}^{-1}\left\{\frac{1}{(1+\omega^2)}\right\} * \mathcal{F}^{-1}\left\{\frac{1}{(1+\omega^2)}\right\} \\ &= \frac{1}{2}e^{-|\tau|} * \frac{1}{2}e^{-|\tau|} \\ &= \frac{1}{4}\int_{-\infty}^{\infty} e^{-|t|}e^{-|\tau-t|}\ dt \\ &= \frac{1}{4}\int_{-\infty}^{0} e^t e^{t-\tau}\ dt + \frac{1}{4}\int_0^{\tau} e^{-t}e^{t-\tau}\ dt + \frac{1}{4}\int_{\tau}^{\infty} e^{-t}e^{\tau-t}\ dt \qquad \text{(for } \tau \geq 0) \\ &= \frac{1}{4}\left[\int_{-\infty}^{0} e^{2t-\tau}\ dt + \int_0^{\tau} e^{-\tau}\ dt + \int_{\tau}^{\infty} e^{t-2\tau}\ dt\right] \\ &= \frac{1}{4}\left[\frac{1}{2}e^{-\tau} + \tau e^{-\tau} + \frac{1}{2}e^{-\tau}\right] \\ &= \frac{1}{4}\left(\tau e^{-\tau} + e^{-\tau}\right) \qquad \text{(for } \tau \geq 0)\end{aligned}$$

Similarly,

$$R_X(\tau) = \frac{1}{4}\left(-\tau e^{\tau} + e^{\tau}\right) \qquad \text{(for } \tau < 0)$$

Therefore,

$$R_X(\tau) = \frac{1}{4}e^{-|\tau|}(|\tau| + 1)$$

8. **Suppose that ω is a random variable with p.d.f. f_ω and θ is a random variable independent of ω uniformly distributed in $(-\pi, \pi)$. Define a random process by $X_t = a\cos(\omega t + \theta),\ t \in \mathbb{R}$ where a is a constant. Find the power spectral density of $\{X_t\}$.**

$$
\begin{aligned}
E[X_{t_1}X_{t_2}] &= E\{a^2\cos(\omega t_1+\theta)\cos(\omega t_2+\theta)\\
&= \frac{1}{2}a^2 E\{\cos(\omega t_1-\omega t_2)-\cos(\omega t_1+\omega t_2+2\theta)\}\\
&= \frac{1}{2}a^2\left[E\{\cos(\omega t_1-\omega t_2)\}-\underbrace{E\{\cos(\omega t_1+\omega t_2+2\theta)\}}_{0}\right]\\
&= \frac{1}{2}a^2 E\{\cos(\omega t_1-\omega t_2)\}\\
&= \frac{1}{2}a^2\int_{-\infty}^{\infty}\cos(\tau\omega)f_\omega(\omega)\ d\omega = \frac{1}{2}a^2\int_{-\infty}^{\infty}\frac{e^{j\omega\tau}+e^{-j\omega\tau}}{2}f_\omega(\omega)\ d\omega\\
&= \frac{1}{4}a^2\ 2\pi[\mathcal{F}^{-1}\{f_\omega(\omega)\}+\mathcal{F}^{-1}\{f_\omega(-\omega)\}]\\
&= \frac{\pi a^2}{2}[\mathcal{F}^{-1}\{f_\omega(\omega)\}+\mathcal{F}^{-1}\{f_\omega(-\omega)\}]
\end{aligned}
$$

Therefore,

$$
S_X(\omega) = \frac{\pi a^2}{2}[f_\omega(\omega)+f_\omega(-\omega)]
$$

9. **Suppose that $\{X_t, t \in \mathbb{R}\}$ is a w.s.s., zero-mean, Gaussian random process with autocorrelation function $R_X(\tau), \tau \in \mathbb{R}$ and power spectral density $S_X(\omega), \omega \in \mathbb{R}$. Define the random process $\{Y_t, t \in \mathbb{R}\}$ by $Y_t = (X_t)^2, t \in \mathbb{R}$. find the mean, autocorrelation, and powerspectral density of $\{Y_t, t \in \mathbb{R}\}$.**

Mean :

$$
\mu_y(t) = E[X_t^2] = R_X(0) = \sigma^2
$$

Autocorrelation :

$$
\begin{aligned}
R_Y(t,s) &= E[Y_tY_s] = E[X_t^2X_s^2]\\
&= \left.\frac{\partial^4}{\partial u^2\partial v^2}\Phi_{X_tX_s}(u,v)\right|_{u=v=0}\frac{1}{i^4}\\
&\ \ \vdots\\
&= R_X^2(0)+2R_X^2(\tau) \qquad (\tau = t-s)
\end{aligned}
$$

PSD :

$$
S_Y(\omega) = \mathcal{F}\{R_Y(\tau)\} = 2\pi R_X^2(0)\delta(\omega) + 2(S_X(\omega) * S_X(\omega))
$$

10. **Suppose U and V are independent random variables with $E[U] = E[V] = 0$ and $\mathrm{var}(U) = \mathrm{var}(V) = 1$. Define random processes by**

$$
X_t = U\cos t + V\sin t \quad Y_t = U\sin t + V\cos t, \quad t \in \mathbb{R}.
$$

Find the autocorrelation and cross-correlation functions of $\{X_t, t \in \mathbb{R}\}$ and $\{Y_t, t \in \mathbb{R}\}$. Are $\{X_t\}$ and $\{Y_t\}$ jointly wide sense stationary? Are they individually wide sense stationary?

$$\begin{aligned}
R_X(t,s) &= E[X_t X_s] = E[(U\cos t + V\sin t)(U\cos s + V\sin s)] \\
&= E[U^2\cos t\cos s + UV(\cos t\sin s + \sin t\cos s) + V^2\sin t\sin s] \\
&= \cos t\cos s E[U^2] + E[U]E[V](\cos t\sin s + \sin t\cos s) + E[V^2]\sin t\sin s \\
&= \cos t\cos s + \sin t\sin s \\
&= \cos(t-s)
\end{aligned}$$

Similarly,

$$R_Y(t,s) = cos(t-s)$$

$$\begin{aligned}
R_{XY}(t,s) &= E[X_t Y_s] = E[(U\cos t + V\sin t)(U\sin s + V\cos s)] \\
&= E[U^2\cos t\sin s + UV(\cos t\cos s + \sin t\sin s) + V^2\sin t\cos s] \\
&= E[U^2]\cos t\sin s + E[UV](\cos t\cos s + \sin t\sin s)E[V^2]\sin t\cos s \\
&= \cos t\sin s + \sin t\cos s \\
&= \sin(t+s)
\end{aligned}$$

$$\mu_X(t) = 0 \qquad \mu_Y(t)$$

So, $\{X_t\}$ and $\{Y_t\}$ are individually WSS, but not jointly WSS.