Utah State University
ECE 6010
Stochastic Processes
Homework # 7
Due Friday Nov. 5, 2004

1. Suppose $\{X_t, t \geq 0\}$ is a homogeneous Poisson process with parameter λ. Define a random variable τ as the time of the first occurrence of an event. Find the p.d.f. and the mean of τ.

2. Suppose $\{X_t, t \in \mathbb{R}\}$ is a w.s.s. random process with autocorrelation function $R_X(\tau)$. Show that if R_X is continuous at $\tau = 0$ then it is continuous for all $\tau \in \mathbb{R}$. (Hint: Use the Schwartz inequality.)

3. Under the conditions of problem 2, show that for $a > 0$,
$$P(|X_{t+\tau} - X_t| \geq a) \leq \frac{2(R_X(0) - R_x(\tau))}{a^2}.$$

4. Suppose A and B are random variables with $E[A^2] < \infty$ and $E[B^2] < \infty$. Define the random processes $\{X_t, t \in \mathbb{R}\}$ and $\{Y_t, t \in \mathbb{R}\}$ by
$$X_t = A + Bt \qquad Y_t = B + At, \qquad t \in \mathbb{R}.$$
Find the mean, autocorrelation, and cross correlations of these random processes in terms of the moments of A and B.

5. Let $p_k(t, s) = X_t - X_s$, where X_t is a homoegenous Poisson counting process with rate λ. Show that the differential equation
$$\frac{\partial}{\partial t} p_k(t, s) = \lambda[p_{k-1}(t, s) - p_k(t, s)]$$
is solved by
$$p_k(t, s) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!} \qquad k = 0, 1, \ldots, \qquad t > s \geq 0.$$

6. Let $p_k(t, s) = X_t - X_s$, where X_t is an inhomoegenous Poisson counting process with time-varying rate λ_t, where λ_t is a continuous nonnegative function. Show that the differential equation
$$\frac{\partial}{\partial t} p_k(t, s) = \lambda_t[p_{k-1}(t, s) - p_k(t, s)]$$

is solved by

$$p_k(t,s) = \frac{e^{-\lambda \int_s^t \lambda_x dx} (\int_s^t \lambda_x \, dx)^k}{k!} \qquad k = 0, 1, \ldots, \qquad t > s \geq 0.$$

Also, determine the mean and autocorrelation functions of $\{X_t, t \in \mathbb{R}\}$.

7. Suppose $\{X_t, t \in \mathbb{R}\}$ is a random process with power spectral density

$$S_X(\omega) = \frac{1}{(1+\omega^2)^2}$$

Find the autocorrelation function of X_t.

8. Suppose that ω is a random variable with p.d.f. f_ω and θ is a random variable independent of ω uniformly distributed in $(-\pi, \pi)$. Define a random process by

$$X_t = a \cos(\omega t + \theta), \qquad t \in \mathbb{R}$$

where a is a constant. Find the power spectral density of $\{X_t\}$.

9. Suppose that $\{X_t, t \in \mathbb{R}\}$ is a w.s.s., zero-mean, Gaussian random process with autocorrelation function $R_X(\tau)$, $\tau \in \mathbb{R}$ and power spectral density $S_X(\omega)$, $\omega \in \mathbb{R}$. Define the random process $\{Y_t, t \in \mathbb{R}\}$ by $Y_t = (X_t)^2$, $t \in \mathbb{R}$. Find the mean, autocorrelation, and power spectral density of $\{Y_t, t \in \mathbb{R}\}$.

10. Suppose U and V are independent random variables with $E[U] = E[V] = 0$ and $\text{var}(U) = \text{var}(V) = 1$. Define random processes by

$$X_t = U \cos t + V \sin T \qquad\qquad Y_t = U \sin t + V \cos t, \qquad t \in \mathbb{R}.$$

Find the autocorrelation and cross-correlation function s of $\{X_t, t \in \mathbb{R}\}$ and $\{Y_t, t \in \mathbb{R}\}$. Are $\{X_t\}$ and $\{Y_t\}$ jointly wide sense stationary? Are they individually wide sense stationary?