Utah State University
ECE 6010
Stochastic Processes
Homework # 6 Solutions

1. Suppose $\{X_n\})_{n=1}^{\infty}$ is a sequence of independent r.v.s each of which is uniformly distributed on the interval $(0,1)$. Define a sequence of r.v.s $\{Z_n\}$ by $Z_n = n(1-Y_n)$, where $Y_N = \max_{1\leq i\leq n} X_i$. Show that $\{Z_n\}_{n=1}^{\infty}$ converges in distribution to an exponential r.v. with p.d.f.

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here,

$$F_{Z_n}(z) = P(Z_n \leq z) = P(n(1-Y_n) \leq z) = P(Y_n \geq (1-z/n)) = 1-P(y_n < (1-z/n))$$

$$= 1-P(\max_{1\leq i\leq n} X_i < (1-z/n)) = 1-P(X_1 < (1-z/n), X_2 < (1-z/n), \ldots, X_n < (1-z/n))$$

$$= 1 - P(X_1 < (1-z/n)) \cdot P(X_2 < (1-z/n)) \cdots P(X_n < (1-z/n))$$

Now,

$$P(X_i < (1-z/n)) = \begin{cases} 0 & (1-z/n) < 0 \text{that is, if } z > n \\ 1-z/n & 0 \leq (1-z/n) \leq 1 \text{that is, if } 0 \leq z \leq n \\ 1 & (1-z/n) > 1 \text{that is, if } z < 0 \end{cases}$$

therefore,

$$F_{Z_n}(z) = \begin{cases} 0 & z > n \\ 1-(1-z/n)^n & 0 \leq z \leq n \\ 1 & z < 0 \end{cases}$$

we have $\lim_{n\to\infty}(1-z/n)^n = e^{-z}$, so

$$\lim_{n\to\infty} F_{Z_n}(z) = F_Z(z) = \begin{cases} 1-e^{-z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

Therefore,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} e^{-z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

So it converges in distribution.

2. Suppose $X_n \to X$ (i.p.) and that there is a constant C such that $|X_n| \leq C$ for all n. Show that $X_n \to X$ (m.s.)

   We have $X_n \to X$ (i.p.) and $|X_n| \leq C$. Therefore,

$$P(|X_n - X| > \varepsilon) \to 0$$

   Define,

$$A = \{|X_n - X| > \varepsilon\} \quad \text{and} \quad B = \{|X_n - X| \leq \varepsilon\}$$

   and let $I_A(x)$ and $I_B(x)$ be the corresponding indicator functions, so that $I_A + I_B = 1$.

$$\begin{aligned}
E(|X_n - X|^2) &= E(|X_n - X|^2(I_A + I_B)) = E(|X_n - X|^2(I_A)) + E(|X_n - X|^2(I_B)) \\
&\leq E(|X_n - X|^2(I_A)) + \varepsilon^2 \qquad \text{(since over } I_B \text{ , } |X_n - X| \leq \varepsilon \text{ and } P(I_B) \to 1) \\
&= E((x_n^2 - 2x_n x + x^2)I_A) + \varepsilon^2 \\
&\leq E(4C^2 I_A) + \varepsilon^2 \;\; (\text{ as } |X_n| \leq C \text{ }) \\
&= 4C^2 E(I_A) + \varepsilon^2 = \varepsilon^2 \qquad (\qquad P(I_A) \to 0)
\end{aligned}$$

   Taking limit $n \to \infty$ we have $\varepsilon \to 0$. Therefore we have,

$$\lim_{n \to \infty} E((X_n - X)^2) \to 0$$

   Therefore, $X_n \to X$ (i.p.) $\Rightarrow X_n \to X$ (m.s.) if $|X_n| \leq C$.

3. Suppose $X_n \to C$ (in distribution), where C is a constant. Show that $X_n \to C$ (i.p.)

   $X_n \to C$ (i.p.) $\Rightarrow P(|X_n - C| > \varepsilon) \to 0$.

$$\begin{aligned}
P(|X_n - C| > \varepsilon) &= P(X_n - C > \varepsilon) + P(X_n - C < -\varepsilon) \\
&= P(X_n > C + \varepsilon) + P(X_n < C - \varepsilon) \\
&= P(X_n > C + \varepsilon) + P(X_n \leq C - \varepsilon) \\
&= 1 - F_{X_n}(C + \varepsilon) + F_{X_n}(C - \varepsilon) \\
&\to 1 - 1 + 0 \qquad \text{(by convergence in distribution)}
\end{aligned}$$

   Therefore,

$$P(|X_n - C| > \varepsilon) = 0 \Rightarrow \lim_{n \to \infty} P(|X_n - C| > \varepsilon) = 0 \Rightarrow X_n \to X \text{ (i.p.)}$$