Utah State University
ECE 6010
Stochastic Processes
Homework # 5 Solutions

1. Let $X_1 \sim \mathcal{U}(0,1)$ and $X_2 \sim \mathcal{U}(0,1)$ (independent). Let $Y_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2)$ and $Y_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2)$. Show that $Y_1 \sim \mathcal{N}(0,1)$ and $Y_2 \sim \mathcal{N}(0,1)$

   Here we have,
$$Y_1^2 + Y_2^2 = -2\ln X_1[(\cos(2\pi X_2))^2 + (\sin(2\pi X_2))^2] = -2\ln X_1$$
   Therefore,
$$X_1 = e^{-\frac{Y_1^2+Y_2^2}{2}}$$
   Also, $Y_2/Y_1 = \tan(2\pi X_2)$, therefore
$$X_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{Y_2}{Y_1}\right)$$
   Jacobian is given by
$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}$$
   Therefore,
$$|J|^{-1} = \ldots = \frac{X_1}{2\pi}$$
   Now,
$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= |J|^{-1} f_{X_1,X_2}(x_1,x_2) \\ &= \frac{X_1}{2\pi} f_{X_1}(x_1) f_{X_2}(x_2) \\ &= \frac{1}{2\pi} e^{-\frac{Y_1^2+Y_2^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{Y_2^2}{2}} \\ &\Rightarrow Y_1 \sim \mathcal{N}(0,1) \text{ and } Y_2 \sim \mathcal{N}(0,1) \end{aligned}$$

2. If X and Y are independent and $Y \sim \mathcal{U}(0,1)$, show that $Z = X + Y$ has density $f_Z(z) = F_x(z) - F_x(z-1)$.

   Here X and Y are independent and $Y \sim \mathcal{U}(0,1)$, therefore we have,
$$F_Z(z) = P(z \le Z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{X,Y}(x,y)dxdy = \int_{-\infty}^{\infty} f_Y(y)dy \int_{-\infty}^{z-y} f_X(z-y)dx$$
   Now,
$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}\int_{-\infty}^{\infty} f_Y(y)dy \int_{-\infty}^{z-y} f_X(z-y)dx = \int_{-\infty}^{\infty} f_Y(y) f_X(z-y)dy$$
   As, $Y \sim \mathcal{U}(0,1)$ the above integral exists from 0 to 1 only, therefore
$$f_Z(z) = \int_0^1 f_X(z-y)dy = -F_X(z-y)|_0^1 = F_X(z) - F_X(z-1)$$

3. Let $X \sim \mathcal{N}(0, \sigma^2)$ and $Y \sim \mathcal{N}(0, \sigma^2)$. Let $Z = X^2 + Y^2$. Show that

$$f_Z(z) = \frac{1}{2\sigma^2} e^{-z/2\sigma^2} u(z)$$

Such a random variable is said to be chi-squared (χ^2) distributed with two degrees of freedom.

$$F_Z(z) = P(Z \leq z) = \int\int_{x^2+y^2 \leq z} f_{XY}(x, y)\, dxdy = \int_0^{\pi} \int_0^{\sqrt{z}} \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, rdrd\theta.$$

Taking derivatives,

$$f_Z(z) = \frac{\sqrt{z}}{2\sqrt{z}\sigma^2} e^{-z/2\sigma^2}$$

4. If $X \in \mathcal{U}(0, 1)$ and $Y = -(\ln(X))/\lambda$, show that Y is exponentially distributed.

We have,

$$Y = -\frac{\ln(X)}{\lambda} \quad \Rightarrow \quad \lambda Y = -(\ln(X)) \quad \Rightarrow \quad X = e^{-\lambda Y}$$

$$\frac{dx}{dy} = -\lambda e^{-\lambda Y} \qquad \text{so,} \qquad \left|\frac{dx}{dy}\right| = \lambda e^{-\lambda Y}$$

So,

$$f_Y(y) = f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right| = f_X(e^{-\lambda Y})\lambda e^{-\lambda Y} = \lambda e^{-\lambda Y} \quad (\text{as } f_X(e^{-\lambda Y}) = 1 \text{ for } 0 < Y < \infty)$$

Therefore, we have

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda Y} & \text{for all } Y > 0 \\ 0 & \text{else} \end{cases}$$

5. Let $g(x)$ be a monotone increasing function and let $Y = g(X)$. Show that

$$F_{XY}(x, y) = \begin{cases} F_X(x) & \text{if } y > g(x) \\ F_Y(y) & \text{if } y < g(x). \end{cases}$$

Suppose that $y > g(x)$. Then

$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = P(X \leq x, X \leq g^{-1}(y)).$$

But when $y > g(x)$, the intersection of $(-\infty, x]$ and $(-\infty, g^{-1}(y)]$ is $(-\infty, x]$. So

$$F_{XY}(x, y) = P(X \leq x) = F_X(x).$$

It works oppositely when $y < g(x)$.

6. Let $Z = aX + bY$. Show that

$$f_Z(z) = \frac{1}{|a|} \int_{-\infty}^{\infty} f_{XY}(\frac{z - by}{a}, y)\, dy.$$

Solution: Introduce the auxiliary variable $W = Y$. Then

$$J = a$$

so

$$f_{ZW}(z, w) = \frac{1}{|a|} f_{XY}(\frac{z - by}{a}, y).$$

Then integrate out w to get $f_Z(z)$.

7. Let X and Y be independent with $f_X(x) = \alpha e^{-\alpha x} u(x)$ and $f_Y(y) = \beta e^{-\beta y} u(y)$. Determine the density of:

(a) $Z = X/Y$.

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{zy} f_{XY}(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{zy} f_X(zy) f_Y(y) dx dy$$

Therefore,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(zy) f_Y(y) |y| dy = \int_0^{\infty} \alpha e^{-\alpha z y} \beta e^{-\beta y} y \; dy = \alpha\beta \int_0^{\infty} e^{-y(\alpha z + \beta)} dy = \ldots = \frac{\alpha\beta}{(\beta + \alpha z)^2}$$

(b) $Z = \max(X, Y)$.

$$F_Z(z) = p(Z \leq z) = P(\max(X, Y) \leq z) = P(X \leq z, Y \leq z) = P(X \leq z) P(Y \leq z)$$

Therefore,

$$F_Z(z) = F_X(z) F_Y(z) = \int_0^z f_X(x) dx \int_0^z f_Y(y) dy = \int_0^z \alpha e^{-\alpha x} dx \int_0^z \beta e^{-\beta y} dy = (1 - e^{-\alpha z})(1 - e^{-\beta z})$$

So,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = -(\alpha + \beta) e^{-z(\alpha+\beta)} + \alpha e^{-\alpha z} + \beta e^{-\beta z}$$

8. Show that if X, Y are i.i.d. $\mathcal{N}(0, \sigma^2)$, then $Z = X/Y$ has a Cauchy distribution,

$$f_Z(z) = \frac{1/\pi}{z^2 + 1}.$$

We have

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2\sigma^2}(x^2 + y^2)\right].$$

Using the result from the notes we have with $Z = X/Y$

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_{XY}(zy, y) \, dy.$$

Then

$$\begin{aligned} f_Z(z) &= \frac{2}{2\pi\sigma^2} \int_0^{\infty} y e^{-y^2(z^2+1)/2\sigma^2} \, dy \\ &= \frac{1}{\pi\sigma^2} \int_0^{\infty} y e^{-y^2 a} \, dy \qquad \text{with } a = (z^2+1)/2\sigma^2 \\ &= \frac{1}{\pi\sigma^2} \int_0^{\infty} -\frac{1}{2a} e^{-u} \, du \qquad \text{use } u = -y^2 a, \text{ so } y dy = -du/2a \\ &= \frac{1}{\pi(z^2+1)}. \end{aligned}$$