Utah State University
ECE 6010
Stochastic Processes
Homework # 4
Due Friday Oct. 7, 2005

1. Suppose $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.
   (a) Show that $E[\mathbf{X}] = \boldsymbol{\mu}$ and $\text{cov}(\mathbf{X}, \mathbf{X}) = \Sigma$.
   (b) Show that $A\mathbf{X} + \mathbf{b} \sim \mathcal{N}(A\boldsymbol{\mu} + \mathbf{b}, A\Sigma A^T)$.
   (c) Suppose $\Sigma > 0$ and write $\Sigma = CC^T$. Show that $C^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim \mathcal{N}(\mathbf{0}, I)$.

2. Suppose you have a random number generator which is capable of generating random numbers distributed as $X \sim \mathcal{N}(0, 1)$. Describe how to generate random vectors $\mathbf{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

3. Suppose X and Y are r.v.s. Show that $E[(X - h(Y))^2]$ is minimized over all functions h when h is the function
$$h(y) = E[X|Y = y].$$
Assume $E[X^2] < \infty$.

4. Let $(X, Y) \sim \mathcal{N}(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$. Let $\mathbf{X} = \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, where
$$\Sigma = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}.$$
Determine the relationship between $\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho$ and $\boldsymbol{\mu}$ and Σ.

5. Suppose $\mathbf{X} = \left[\begin{smallmatrix} X_1 \\ X_2 \\ X_3 \end{smallmatrix}\right] \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ where
$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 8 \end{bmatrix}$$
   (a) The value $X_1 = 1.5$ is measured. Determine the best estimate for (X_2, X_3).
   (b) In a separate problem, the values $X_2 = 1$ and $X_3 = 5$ are measured. Determine the best estimate of X_1.

(c) Determine a random vector $\mathbf{Y}$ which is a whitened version of $\mathbf{X}$.

Problems from Grimmet & Stirzaker

1. Ex 3.7.5. What is requested is $E[T-t|T>t]$, i.e., the mean subsequent lifetime given that the machine is still running after t days. Then use the hint from the book. Note that in (a), $P(T>t) = \frac{1}{N+1}(N-t)$.

2. Ex 3.7.7. Hint: Show that $P(\text{robot faulty}|\text{fault not detected}) = \frac{\phi(1-\delta)}{1-\phi\delta} \triangleq \pi$. Hence argue that the number of faulty passed robots, given Y, is distributed as $\mathcal{B}(n-Y,\pi)$, which has mean $(n-Y)\pi$. Hence show that $E[X|Y] = Y + (n-Y)\pi$.

3. Ex 4.1.1(a)

4. Ex 4.1.2

5. Ex 4.2.1. Hint: think geometric r.v.

6. Ex 4.2.2. Hint: $P(\max(X,Y) \leq v) = P(X \leq v, Y \leq v)$.

7. Ex 4.4.1. Hint: integrate by parts.

8. Ex 4.6.4(b)