ECE 6010 Stochastic Processes Homework #3

Problems from Grimmet & Stirzaker:

1. Prob 2.7.4

a)

$$P(\frac{1}{2} < x \leq \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}) = \frac{1}{2}.$$

b)

$$P(1 < X < 2) = F(2) - F(1) = \frac{1}{2}.$$

c)

$$P(Y \leq X) = P(X^2 \leq X) = P(X \leq 1) = F(1) = \frac{1}{2}.$$

d)

$$P(X \leq 2Y) = P(X \leq 2X^2) = P(X \geq \frac{1}{2}) = \frac{3}{4}.$$

e)

$$P(X + Y \leq \frac{3}{4}) = P(X + X^2 \leq \frac{3}{4}) = P(X \leq \frac{1}{2}) = \frac{1}{4}$$

f)

$$P(\sqrt{X} \leq Z) = P(X \leq Z^2) = \frac{1}{2}z^2$$

if $0 \leq z \leq \sqrt{2}$.

2.Prob 2.7.7. Let T be the numbers of people on given typical flights of TWA, and B be the numbers of people on given typical flights of BA.

$$P(T = k) = \binom{10}{k} \left(\frac{9}{10}\right)^k \left(\frac{1}{10}\right)^{10-k},$$

$$P(B = k) = \binom{20}{k} \left(\frac{9}{10}\right)^k \left(\frac{1}{10}\right)^{20-k}.$$

Now P(TWA overbooked)= P(T=10)=$(\frac{9}{10})^{10}$

P(BA overbooked) $= P(B \geq 19) = 20(\frac{9}{10})^{19}(\frac{1}{10}) + (\frac{9}{10})^{20}$, of which the latter is the larger.

3. Prob 2.7.9. (a)

$$P(X^+ \le x) = \begin{cases} 0 & \text{if } x < 0, \\ F(x) & x \ge 0 \end{cases}$$

(b)

$$P(X^- \le x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - lim_{y \uparrow -x} F(y) & \text{if } x \ge 0 \end{cases}$$

(c) $P(|X| \le x) = P(-x \le X \le x)$, if $x \ge 0$. Therefore.

$$P(|X| \le x) = \begin{cases} 0 & \text{if } x < 0, \\ F(x) - lim_{y \uparrow -x} F(y) & \text{if } x \ge 0 \end{cases}$$

(d)

$$P(-X \le x) = 1 - lim_{y \uparrow -x} F(y).$$

4. Ex 3.3.1. (a) No!
(b) Let X have mass function: $f(-1) = \frac{1}{9}, f(\frac{1}{2}) = \frac{4}{9}, f(2) = \frac{4}{9}$. Then $E(X) = -\frac{1}{9} + \frac{2}{9} + \frac{8}{9} = 1 = -\frac{1}{9} + \frac{2}{9} + \frac{8}{9} = E(1/X)$.

5. Ex 3.4.1. Let I_j be the indicator function of the event that the outcome of the $(i{+}1)$th toss is different from the outcome of the jth toss. The number R of distinct runs is given by $R = 1 + \sum_{j=1}^{n-1} I_j$. Hence

$$E[R] = 1 + (n-1)E[I_1] = 1 + (n-1)2pq,$$

where $q = 1 - p$. Now remark that I_j and I_k are independent if $|j - k| > 1$, so that

$$E\{(R-1)^2\} = E\left\{(\sum_{j=1}^{n-1} I_j)^2\right\} = (n-1)E[I_1] + 2(n-2)E[I_1 I_2]$$

$$+\{(n-1)^2 - (n-1) - 2(n-2)\}E[I_1]^2.$$

Now $E[I_1^2] = E[I_1] = 2pq$ and $E[I_1 I_2] = p^2 q + pq^2 = pq$, and therefore

$$var(R) = var(R-1) = (n-1)E[I_1] + 2(n-2)E[I_1 I_2] - \{(n-1) + 2(n-2)\}E[I_1]^2$$

$$= 2pq(2n - 3 - 2pq(3n - 5)).$$

6. Ex 3.4.2. The required total is $T = \sum_{i=1}^{k} X_i$, where X_i is the number shown on the ith ball. Hence $E[T] = kE[X_1] = \frac{1}{2}k(n+1)$

$$E[T^2] = E\left\{(\sum_{i=1}^{k} X_i)^2\right\} = kE[X_1^2] + k(k-1)E[X_1X_2]$$

$$= \frac{k}{n}\sum_{1}^{n} j^2 + \frac{k(k-1)}{n(n-1)}\sum_{i\neq j} ij$$

$$= \frac{1}{6}k(n+1)(2n+1) + \frac{1}{12}k(k-1)(3n+2)(n+1).$$

Hence

$$Var(T) = E[T^2] - E[T]^2 = \frac{1}{12}(n+1)k(n-k)$$

7. Ex 3.5.2. The total number H of heads satisfies

$$P(H = X) = \sum_{n=x}^{\infty} P(H = x|N = n)P(N = n)$$

$$= \sum_{n=x}^{\infty} \left(\begin{array}{c} n \\ x \end{array}\right) p^x(1-p)^{n-x}\frac{\lambda^n e^{-\lambda}}{n!}$$

$$= \frac{(\lambda p)^x e^{-\lambda p}}{x!}\sum_{n=x}^{\infty}\frac{\{\lambda(1-p)\}^{n-x}e^{-\lambda(1-p)}}{(n-x)!}.$$

The last summation equals 1, since it is the sum of the values of the Poission mass function with parameter $\lambda(1-p)$.

8. Ex 3.6.5. (a) $log y \leq y-1$ with equality if and only if $y = 1$. Therefore,

$$E\left[log\frac{f_Y(X)}{f_X(X)}\right] \leq E\left[\frac{f_Y(X)}{f_X(X)} - 1\right] = 0,$$

with equality if an only if $f_Y = f_X$.

(b)

$$E\left[log\frac{f(x,y)}{f_X(x)f_Y(y)}\right] \leq E\left[\frac{f(x,y)}{f_X(x)f_Y(y)} - 1\right]$$

$$-I = E\left[log\frac{f_X(x)f_Y(y)}{f_{XY}(x,y)} - 1\right] \leq E\left[\frac{f_X(x)f_Y(y)}{f_{XY}(x,y)} - 1\right] = 0$$

so $I \geq 0$