Utah State University
ECE 6010
Stochastic Processes
Homework # 3 Solutions

1. Show that $\text{cov}(aX+b, cY+d) = ac\,\text{cov}(X,Y)$.

$$\text{cov}(aX+b, cY+d) = E\left[(aX+b-E[aX+b])\,(cY+d-E[cY+d])\right]$$
$$E[aX+b] = a\mu_x + b \qquad \& \qquad E[cY+d] = c\mu_y + d$$

Therefore,

$$\text{cov}(aX+b, cY+d) = E[a(X-\mu_x)\cdot c(Y-\mu_y)] = ac\,E[(X-\mu_x)(Y-\mu_y)] = ac\,\text{cov}(X,Y)$$

2. Suppose $X \sim \mathcal{N}(0,\sigma^2)$. Use the ch.f. of X to find an expression for $E[X^n]$, $n \in \mathbb{Z}^+$.

$$\phi_X(u) = e^{j\mu u - \frac{1}{2}u^2\sigma^2} = e^{\frac{1}{2}u^2\sigma^2}$$
$$E[X^n] = i^{-n}\frac{d^n}{du^n}\phi_X(u)\Big|_{u=0} = i^{-n}\frac{d^n}{du^n}e^{\frac{1}{2}u^2\sigma^2}\Big|_{u=0} = \frac{d^n}{du^n}i^{-n}\left(1-\frac{\sigma^2u^2}{2^1 1!}+\frac{\sigma^4u^4}{2^2 2!}-\frac{\sigma^6u^6}{2^3 3!}+\cdots\right)\Big|_{u=0}$$
$$= \begin{cases} 0 & n \text{ odd} \\ \frac{\sigma^n n!}{2^{n/2}(n/2)!} & n \text{ even} \end{cases} = \begin{cases} 0 & n \text{ odd} \\ 1\cdot 3\cdot 5\ldots(n-1)\sigma^n & n \text{ even} \end{cases}$$

3. Suppose X and Y are the indicator functions of events A and B, respectively. Find $\rho(X,Y)$, and show that X and Y are independent if and only if $\rho(X,Y)=0$.

$$X = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \qquad Y = \begin{cases} 1 & y \in B \\ 0 & y \notin B \end{cases}$$
$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{E[X,Y]-E[X]E[Y]}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{P(x\in A \text{ and } y \in B) - P(x\in A)P(y\in B}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

So from the equation above,

$$\rho = 0 \quad \Rightarrow \quad P(x\in A \text{ and } y\in B) = P(x\in A)P(y\in B) \quad \Rightarrow \quad X, Y \text{are independent}$$
$$X, Y \text{ independent} \quad \Rightarrow \quad P(x\in A \text{ and } y\in B) = P(x\in A)P(y\in B) \quad \Rightarrow \quad \rho = 0$$

Therefore, $\rho(X,Y) = 0 \Leftrightarrow X$ and Y are independent.

4. Suppose $\phi(u)$ is a ch.f. Show that $|\phi(u)|^2$ is also a ch.f.

$$\phi(u) = E[e^{jux}] = \int_{-\infty}^{\infty} e^{jux} f_X(x)dx$$

$$|\phi(u)|^2 = \phi(u) \cdot \phi^*(u) = E[e^{jux}]E[e^{-jux}] = \int_{-\infty}^{\infty} e^{jux} f_X(x)dx \int_{-\infty}^{\infty} e^{-jux} f_X(x)dx$$

Let $Y = -X$ be another random variable, so $f_Y(y) = f_X(-x)$ and $dx = -dy$.

$$|\phi(u)|^2 = \int_{-\infty}^{\infty} e^{jux} f_X(x)dx \int_{-\infty}^{\infty} e^{juy} f_Y(y)dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ju(x+y)} f_X(x)f_Y(y)dxdy = E[e^{ju(x+y)}]$$

So, $|\phi(u)|^2$ is a ch.f. for r.v. $X + Y$.

5. Suppose X and Y are jointly Gaussian. Use ch.f.s to show that $\rho(X, Y) = \rho$.

$$\phi_{X,Y}(u, v) = \exp\left[i(u\mu_x + v\mu_y) - \frac{1}{2}(u^2\sigma_x^2 + v^2\sigma_y^2 + 2uv\mu_x\mu_y\rho)\right]$$

$$E[XY] = -\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi(u, v)\bigg|_{u,v=0} = \cdots = \sigma_x\sigma_y\rho + \mu_x\mu_y$$

Now,

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{E[X, Y] - E[X]E[Y]}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{\sigma_x\sigma_y\rho + \mu_x\mu_y - \mu_x\mu_y}{\sigma_x\sigma_y} = \rho$$

6. Suppose X and Y are jointly continuous. (a) Show that

$$F_{Y|X}(b|x) = \int_{-\infty}^{b} \frac{f_{XY}(x, y)}{f_X(x)} dx$$

and thus that

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

For X and Y jointly continuous,

$$F_{Y|X}(b|x) = P(Y \leq b|X = x) = \frac{P(Y \leq b \text{ and } X = x)}{P(X = x)} = \frac{\int_{-\infty}^{b} f_{XY}(x, y)dy}{f_X(x)} = \int_{-\infty}^{b} \frac{f_{XY}(x, y)}{f_X(x)}dy$$

Now,

$$f_{Y|X}(y|x) = \frac{\partial}{\partial y} F_{Y|X}(y|x) = \frac{\partial}{\partial y} \int_{-\infty}^{y} \frac{f_{XY}(x,y)}{f_X(x)} dy = \frac{f_{XY}(x,y)}{f_X(x)}$$

(b) Suppose $\int_{-\infty}^{\infty} |y| f_{Y|X}(y|x)\, dy < \infty$. Show that $E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$.

$$E[Y|X=x] = \int_{-\infty}^{\infty} y \; dF_{Y|X}(y|x)$$

Now,

$$F_{Y|X}(y|x) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) dy$$

therefore,

$$dF_{Y|X}(y|x) = f_{Y|X}(y|x) dy$$

substituting this in the first equation we get

$$E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy$$

7. Suppose X and Y are independent continuous r.v.s with c.d.f.s F_X and F_Y, respectively. Suppose further that $F_X(b) \geq F_Y(b)$ for all $b \in \mathbb{R}$. Show that $P(X \geq Y) \leq 1/2$.

$$\begin{aligned}
P(X \geq Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x,y) dx dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_X(x) f_Y(y) dx dy \text{ (because independent)} \\
&= \int_{-\infty}^{\infty} F_Y(x) f_X(x) dx \\
&\leq \int_{-\infty}^{\infty} F_X(x) f_X(x) dx = \left. \frac{F_X^2(x)}{2} \right|_{-\infty}^{\infty} = 1/2 - 0 = 1/2
\end{aligned}$$

8. Prove Jensen's inequality for the case of simple-function r.v.'s

   First, we prove that the convexity idea generalizes to multiple points. For a convex function $c(x)$ we know that

$$\lambda c(x_1) + (1-\lambda) c(x_2) \geq c(\lambda x_1 + (1-\lambda) x_2).$$

The general result is:

$$\sum_{i=1}^{n} \lambda_i c(x_i) \geq c(\sum_{i=1}^{n} \lambda_i x_i) \tag{***}$$

where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$.

We'll do it for three points, from which the induction to n points should be straightforward. Let $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Consider

$$\lambda_1 c(x_1) + \lambda_2 c(x_2) + \lambda_3 c(x_3) = (\lambda_1 c(x_1) + \lambda_2 c(x_2)) + \lambda_3 c(x_3)$$

Factor out of the first two terms the quantity $\lambda_1 + \lambda_2$:

$$(\lambda_1 + \lambda_2)\left[\frac{\lambda_1}{\lambda_1 + \lambda_2} c(x_1) + \frac{\lambda_2}{\lambda_1 + \lambda_2} c(x_2)\right] + \lambda_3 c(x_3).$$

Since $\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} = 1$, convexity applies to the two terms in the square brackets:

$$(\lambda_1+\lambda_2)\left[\frac{\lambda_1}{\lambda_1 + \lambda_2} c(x_1) + \frac{\lambda_2}{\lambda_1 + \lambda_2} c(x_2)\right] + \lambda_3 c(x_3) \geq (\lambda_1+\lambda_2) c(\frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2) + \lambda_3 c(x_3). \tag{*}$$

Now letting $x^* = \frac{\lambda_1}{\lambda_1+\lambda_2} x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} x_2$ and $\lambda^* = \lambda_1 + \lambda_2$, we see that we have obtained

$$\lambda^* c(x^*) + (1 - \lambda^*) c(x_3)$$

for which convexity applies again:

$$\lambda^* c(x^*) + (1 - \lambda^*) c(x_3) \geq c(\lambda^* x^* + (1 - \lambda^* x_3). \tag{**}$$

Combining (*) and (**) we obtain

$$\lambda_1 c(x_1) + \lambda_2 c(x_2) + \lambda_3 c(x_3) \geq c(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3).$$

Now to Jensen's inequality. It, too, is proved by induction. We will demonstrate explicitly the first couple of steps. Suppose $X = b_1 I_{A_i}(\omega)$ (a simple function involving a single set $A_1 \subset \Omega$. Then X takes on two values: b_1, with probability $P(A_1)$ and 0, with probability $P(A_1^c)$. Then

$$E[X] = b_1 P(A_1) + 0P(A_1^c)$$

and for a convex function c

$$E[c(X)] = c(b_1)P(A_1) + c(0)P(A_1^c) \geq c(b_1 P(A_1) + 0P(A_1)^c),$$

where the inequality follows since c is convex, and $P(A_1) + P(A_1^c) = 1$.

Now consider a simple function involving two disjoint sets:

$$X = b_1 I_{A_1}(\omega) + b_2 I_{A_2}(\omega), \qquad A_1 \cap A_2 = \emptyset.$$

Then X takes on three values, b_1, b_2, and 0, with probabilities $P(A_1), P(A_2)$, and $P(A_1^c \cap A_2^c)$, respectively. Then

$$E[X] = b_1 P(A_1) + b_2 P(A_2) + 0P(A_1^c \cap A_2^c).$$

And

$$E[c(X)] = c(b_1)P(A_1)+c(b_2)P(A_2)+c(0)P(A_1^c \cap A_2^c) \geq c(b_1 P(A_1)+b_2 P(A_2)+0P(A_1^c \cap A_2^c)$$

by the convexity in (***).

9. Prove the Schwartz inequality.

Consider the quantity $E[(X - \alpha Y)^2]$, which is ≥ 0 for all values of the real constant α. Expanding, we have

$$0 \leq E[X^2] - 2\alpha E[XY] + \alpha^2 E[Y^2] \tag{*}$$

Now find the value of α that minimizes the right-hand side by differentiating with respect to α:

$$-2E[XY] + 2\alpha E[Y^2] = 0$$

so the minimizing $\alpha = E[XY]/E[Y^2]$. Substitution into (*) we have

$$0 \leq E[X^2] - 2E[XY]^2/E[Y^2] + (E[XY]^2/E[Y^2]^2)E[Y^2].$$

Simplifying, we obtain the expression

$$E[X^2]E[Y^2] \geq E[XY].$$