Utah State University
ECE 6010
Stochastic Processes
Homework # 3
Due Friday Sept. 23, 2005

1. Suppose X and Y are the indicator functions of events A and B, respectively. Find $\rho(X,Y)$, and show that X and Y are independent if and only if $\rho(X,Y)=0$.

2. Suppose $\phi(u)$ is a ch.f. Show that $|\phi(u)|^2$ is also a ch.f.

3. Suppose X and Y are jointly Gaussian. Use ch.f.s to show that $\rho(X,Y)=\rho$.

4. Suppose X and Y are jointly continuous. (a) Show that

$$F_{Y|X}(b|x) = \int_{-\infty}^{b} \frac{f_{XY}(x,y)}{f_X(x)}\,dy$$

   and thus that

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

   (b) Suppose $\int_{-\infty}^{\infty} |y| f_{Y|X}(y|x)\,dy < \infty$. Show that $E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy$.

5. Suppose X and Y are independent continuous r.v.s with c.d.f.s F_X and F_Y, respectively. Suppose further that $F_X(b) \geq F_Y(b)$ for all $b \in \mathbb{R}$. Show that $P(X \geq Y) \leq 1/2$.

6. Prove Jensen's inequality for the case of simple-function r.v.'s

7. Prove the Schwartz inequality.

Problems from **Grimmet & Stirzaker**:

1. Prob 2.7.4

2. Prob 2.7.7. Hint: binomial distribution

3. Prob 2.7.9. Hint:

$$P(X^- \leq x) = \begin{cases} 0 & x < 0, \\ 1 - \lim_{y\uparrow -x} F(y) & x \geq 0. \end{cases}$$

4. Ex 3.3.1. Hint: Let $p_X(-1) = \frac{1}{9}$, $p_X(\frac{1}{2}) = \frac{4}{9}$ and $p_X(2) = \frac{4}{9}$.

5. Ex 3.4.1. Let I_j be the indicator function of the event that the outcome of the $(j+1)$st toss is different from the outcome of the jth toss. The number R of distinct runs is $R = 1 + \sum_{j=1}^{n-1} I_j$. Observe that I_j and I_k are independent if $|j-k| > 1$. Show that

$$E[(R-1)^2] = (n-1)E[I_1] + 2(n-2)E[I_1 I_2] + ((n-1)^2 - (n-1) - 2(n-2))E[I_1]^2.$$

Show that $E[I_1 I_2] = p^2 q + pq^2 = pq$.

6. Ex. 3.4.2. Hint: Let $T = \sum_{i=1}^{k} X_i$, where X_i is the number on the ith ball. Show that: $E[T] = \frac{1}{2}k(n+1)$. show that $E[T^2] = \frac{1}{6}k(n+1)(2n+1) + \frac{1}{12}k(k-1)(3n+2)(n+1)$. Hint:

$$\sum_{k=1}^{N} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

7. Ex 3.5.2. Hint: $P(H = x) = \sum_{n=x}^{\infty} P(H = x | N = n) P(N = n)$.

8. Ex 3.6.5. Hint: $\log y \le y - 1$, with equality if and only if $y = 1$.