7.

$$cov(aX+b, cY+d) = accov(X,Y)$$
$$cov(X,Y) = E[(X-E[X])(Y-E[Y])]$$
$$= E[((aX-b)-E(aX+b))((cY+d)-E(cY+d))]$$
$$E(aX+bY) = aE[X]+bE[Y]$$
$$cov(aX+b, cY+d) = (aX-b-aE[X]+b)(cY+d-cE[Y]+d)$$
$$E[(aX-a\mu_x)(cY-c\mu_y)] = E[a(x-\mu_x)c(Y-\mu_y)]$$
$$cov(aX+b, cY+d) = ac\underbrace{E[(X-\mu_x)(Y-\mu_y)]}_{cov(X,Y)} = accov(X,Y)$$

Q.E.D

8.

$$X \sim \mathcal{N}(0, sigma^2)$$
$$\Phi_X(\mu) = E[e^{i\mu x}] = \int_{-\infty}^{\infty} e^{i\mu x} f_x(x)dx$$
$$f_x(x) = \frac{1}{\sqrt{2\pi^2}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$
$$\Phi_x(\mu) = e^{i\mu u} - \frac{1}{2}u^2\sigma^2 = e^{-\frac{1}{2}u^2\sigma^2}$$
$$E[X^k] = i^{-k}\frac{d^k}{du^k}\Phi_x(u)\mid_{u=0}$$
$$= i^{-k}\frac{d^k}{du^k}e^{-\frac{1}{2}u^2\sigma^2}\mid_{u=0}$$
$$= \frac{d^k}{du^k}i^{-k}(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}...)\mid_{u=0}$$
$$= \frac{d^k}{du^k}i^{-k}(1-\frac{\sigma^2u^2}{2^1 2!}+\frac{\sigma^4u^4}{2^2 2!}-\frac{\sigma^6u^6}{2^3 3!}+\frac{\sigma^8u^8}{2^4 4!}...)\mid_{u=0}$$
$$E[x^k] = \begin{cases} 0 & \text{when k is odd} \\ \frac{\sigma^k k!}{2^{\frac{k}{2}}(\frac{k}{2})!} & \text{when k is even} \end{cases}$$

1.4.4 Solution: The final calculation of $\frac{2}{3}$ refers not to a *single* draw of one ball from an urn containing three, but rather to a composite experiment comprising more than one stage. While it is true that {two black, one white } is the only fixed collection of balls for which a random choice is black with probability $\frac{2}{3}$, the composition of the urn is *not determined* prior to the final draw.

After all, if Carroll's argument were correct then it would apply also in the situation when the urn originally contains just one ball, either black or white. The final probability is now $\frac{3}{4}$, implying that the original ball was one half black and one half white! Carroll was himself aware of the fallacy in this argument.

1.4.5 Solution: (a) (i)

$$P(C_3 \mid G) = \frac{P(C_3 \cap G \mid C_1)P(C_1) + P(C_3 \cap G \mid C_1^c)P(C_1^c)}{P(G \mid C_1)P(C_1) + P(G \mid C_1^c)P(C_1^c)}$$
$$= \frac{0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3}}{1 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3}} = \frac{2}{3}$$

(ii).

$$P(C_3 \mid B) = \frac{P(C_3 \cap B \mid C_1)P(C_1) + P(C_3 \cap B \mid C_1^c)P(C_1^c)}{P(B \mid C_1)P(C_1) + P(B \mid C_1^c)P(C_1^c)}$$
$$= \frac{0 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3}}{b \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3}} = \frac{1}{1+b}.$$

(iii).

$$P(C_3 \mid G) = \frac{P(C_3 \cap G \mid C_1)P(C_1) + P(C_3 \cap G \mid C_1^c)P(C_1^c)}{P(G \mid C_1)P(C_1) + P(G \mid C_1^c)P(C_1^c)}$$
$$= \frac{0 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3}}{1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3}} = \frac{1}{2}.$$

(b) Let $\alpha \in [\frac{1}{2}, \frac{2}{3}]$, and suppose the presenter possesses a coin which falls with heads upward with probability $\beta = 6\alpha - 3$. He flips the coin before the show, and adopts strategy (i) if and only if the coin shows heads, and otherwise strategy(iii). The probability in question is now

$$\frac{2}{3}\beta + \frac{1}{2}(1 - \beta) = \alpha.$$

You never lose by swapping, but whether you gain depends on the presenter's protocol.
(c) Let D denote the first door chosen, and consider the following protocols:
(iv) If D conceals a goat, open it. Otherwise open one of the other two doors at random. In this case $p = 0$.
(v) If D conceals a car, open it. Otherwise open the unique remaining door which conceals a goat. In this case $p = 1$.

1.5.1 Solution:

$$P(A^c \cap B) = P(B \setminus \{A \cap B\}) = P(B) - P(A \cap B)$$
$$= P(B) - P(A)P(B) = P(A^c)P(B).$$

$$P(A^c \cap B^c) = P(A^c \setminus \{B \cap A^c\}) = P(A^c) - P(B \cap A^c)$$
$$= (1 - P(B))(1 - P(A)) = P(A^c)P(B^c).$$

1.5.2 Solution: Suppose $i < j$ and $m < n$. If $j < m$, then A_{ij} and A_{mn} are determined by distinct independent rolls, and are therefore independent.
For the case $j = m$ we have that

$P(A_{ij} \cap A_{jn})$=P(ith, jth, and nth rolls show same number)
$= \sum_{r=1}^{6}$ P(jth and nth rolls both show $r \mid ith$ shows $r) = \frac{1}{36} = P(A_{ij})P(A_{jn})$

so we have pair-wise independence.
But $if i \neq j \neq k$

$$P(A_{ij} \cap A_{jn} \cap A_{ik}) = \frac{1}{36} \neq \frac{1}{216} = P(A_{ij})P(A_{jn})P(A_{ik}).$$

Therefore not independence.

1.5.7 Solution:

(a)

$$P(A) = P(BBB \cup GGG) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$P(B) = P(BGG \cup GBG \cup GGB \cup GGG) = 4 \cdot \frac{1}{8}$$

$$P(C) = \frac{3}{4}$$

$$P(A \cap B) = P(GGG) = \frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2} = P(A)P(B)$$

$$P(B \cap C) = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = P(B)P(C)$$

(b)

$$P(A \cap C) = 0 \neq P(A)P(C)$$

(c) Only in the trivial cases when children are either almost surely boys or almost surely girls.

(d) No.

1.8.5 Solution:

$$P(A \triangle B) = P((A \cup B) \setminus P(A \cap B)) = P(A \cup B) - P(A \cap B)$$
$$= P(A) + P(B) - 2P(A \cap B).$$

1.8.6 Solution:

$$P(A \cup B \cup C) = P((A^c \cap B^c \cap C^c)^c)$$

$$= 1 - P(A^c \cap B^c \cap C^c)$$

$$= 1 - P(A^c \mid B^c \cap C^c)P(B^c \mid C^c)P(C^c)$$

1.8.19 Solution:

(d)

$$P(A \leftrightarrow D \mid AD^c) = P(A \leftrightarrow D \mid AD^c \cap BC^c)p + P(A \leftrightarrow D \mid AD^c \cap BC)(1-p)$$

$$= \{1 - (1 - (1-p)^2)^2\}p + (1-p^2)^2(1-p).$$

(c)

$$P(A \leftrightarrow D \mid BC^c) = P(A \leftrightarrow D \mid AD^c \cap BC^c)p + P(A \leftrightarrow D \mid BC^c \cap AD)(1-p)$$

$$= \{1 - (1 - (1-p)^2)^2\}p + (1-p).$$

(b)

$$P(A \leftrightarrow D \mid AB^c) = P(A \leftrightarrow D \mid AB^c \cap AD^c)p + P(A \leftrightarrow D \mid AB^c \cap AD)(1-p)$$

$$= (1-p)\{1 - (1 - (1-p)^2)^2\}p + (1-p).$$

(a)

$$P(A \leftrightarrow D \mid AB^c) = P(A \leftrightarrow D \mid AD^c)p + P(A \leftrightarrow D \mid AD)(1-p)$$

$$= \{1 - (1 - (1-p)^2)^2\}p^2 + (1-p^2)^2 p(1-p) + (1-p).$$

1.8.20 Solution: We condition on the reuslt of the first toss. If this is a head, then we require an odd number of heads in the next $n-1$ tosses. Similarly, if the first toss is a tail, we require an even number of heads in the next $n-1$ tosses.

Hence p_n=Prob. of even hands after n tosses
=P(even number of n-1 tosses)$\cdot$ P(tails on nth)
+P(odd number of n-1 tosses)$\cdot$ P(heads on nth)
$=(1-P)P_{n-1}+P(1-P_{n-1})$ with $p_0=1$.

As an alternative to induction, we may seek a solution of the form $p_n = A + B\lambda^{n-1} + p$ Hence $A=\frac{1}{2}, B=\frac{1}{2}, \lambda = 1-2p$

1.8.30 In general , there are 365^m different combinations. $\frac{365!}{(365-m)!}$ ways of having different birthdays, $\frac{365!}{(365-m)!365^m}$ probability of being all different, $1-\frac{365!}{(365-m)!365^m}$ of two of them are the same. let $m=23$

$$1-\frac{(365)(364)...(365-23)}{365^{23}}=0.5073$$