Utah State University
ECE 6010
Stochastic Processes
Homework # 2 Solutions

1. Suppose X is a r.v. with c.d.f. F_X. Prove the following:

   (a) F_X is nondecreasing.
   Let $b > a$.

   $$F_X(b) - F_X(a) = P(X \le b) - P(X \le a) = P(X \le a) + P(a < X \le b) - P(X \le a)$$
   $$= P(a < X \le b) \ge 0.$$

   So $F_X(b) \ge F_X(a)$ for $b > a$, which means F_X is nondecreasing.

   (b) $\lim_{a \to \infty} F_X(a) = 1$.

   $$\lim_{a \to \infty} F_X(a) = \lim_{a \to \infty} P(X \le a) = P(\{\omega : X(\omega) \le a\}) = P(\Omega) = 1.$$

   (c) $\lim_{a \to -\infty} F_X(a) = 0$.

   $$\lim_{a \to -\infty} F_X(a) = \lim_{a \to -\infty} P(\{\omega : X(\omega) \le a\}) = P(\emptyset) = 1.$$

   (d) F_X is right continuous.
   Let $B_n = \{\omega \in \Omega : X(\omega) \le a + 1/n\}$ for $n = 1, 2, \ldots$. Note that this is a nested sequence, $B_1 \supset B_2 \supset \cdots$. We have

   $$\lim_{n \to \infty} F_X(a + 1/n) = \lim_{n \to \infty} P(B_n) = P(\lim_{n \to \infty} B_n)$$

   by continuity of probability. But $lim_{n \to \infty} B_n = \{\omega : X(\omega) \le a\}$, so

   $$\lim_{n \to \infty} F_X(a + 1/n) = P(X \le a).$$

   Since the limit from the right is equal to the limiting value, we have right continuity.

   (e) $P(a < X \le b) = F_X(b) - F_X(a)$ if $b > a$.
   $P(a < X \le b) = P(X \le b) - P(X \le a) = F_X(b) - F_X(a)$.

(f) $P(X = a) = F_X(a) - \lim_{b \to a^-} F_X(b)$.
$P(X = a) = P(X \leq a) - P(X < a) = F_X(a) - \lim_{b \to a^-} F_X(b)$.

Also, find expressions for $P(a \leq X \leq b)$, $P(a \leq X < b)$ and $P(a < X < b)$ in terms of F_X.

$$P(a \leq X \leq b) = P(a < X \leq b) + P(X = a) = F_X(b) - F_X(a) + (F_X(a) - \lim_{b \to a^-} F_X(b)).$$

$$\begin{aligned} P(a \leq X < b) &= P(a < X \leq b) + P(X = a) - P(X = b) \\ &= F_X(b) - F_X(a) + (F_X(a) - \lim_{c \to a^-} F_X(c)) - (F_X(b) - \lim_{c \to b^-} F_X(c)) \end{aligned}$$

2. Show that the following are valid p.m.f.s:

(a) Binomial: $f_X(a) = n!/((n-a)!a!)\pi^a(1-\pi)^{n-a}$ if $a \in \{0, 1, \ldots, n\}$.
Need to show that $\sum_a f_X(a) = 1$. Use the binomial theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

with $x = \pi$ and $y = 1 - \pi$. Then

$$\sum_{i=0}^{n} f_X(i) = \sum_{i=0}^{n} \binom{n}{i} \pi^i (1-\pi)^{n-i} = (\pi + 1 - \pi)^n = 1^n = 1.$$

(b) Poisson: $f_X(a) = e^{-\lambda}\lambda^a/a!$ for $a \in \{0, 1, \ldots\}$.
Need to show that $\sum_a f_X(a) = 1$.

$$\sum_{i=0}^{\infty} f_X(i) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1.$$

3. **Find the mean and variance of X when X is**

(a) $\mathcal{N}(\mu, \sigma^2)$**;**
Let $Z \sim \mathcal{N}(0, 1)$, therefore

$$E(Z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-z^2/2} dz$$

Function $ze^{-z^2/2}$ has odd symmetr. Integrating an odd function on a symmetric interval -a,a gives zero. Thus,

$$E(Z) \;=\; \lim_{a\to\infty}\int_{-a}^{a}\frac{1}{\sqrt{2\pi}}ze^{-z^2/2}dz \;=\; 0.$$

For variance,

$$Var(Z) \;=\; \int_{-\infty}^{\infty}(z-E(Z))^2 f(z)dz \;=\; \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}z^2e^{-z^2/2}dz$$

With $u = z$ and $dv = ze^{-z^2/2}dz$, we have $du = dz$ and $v = -e^{-z^2/2}$.

$$Var(Z) = \underbrace{-\frac{1}{\sqrt{2\pi}}ze^{-z^2/2}\Big|_{-\infty}^{\infty}}_{0} \;+\; \underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz}_{1} \;=\; 1$$

Now $X \sim \mathcal{N}(\mu,\sigma^2)$. So, $X = \mu + \sigma Z$.

$$E(X) = E(\mu+\sigma Z) = \mu + \sigma E(Z) = \mu$$

$$Var(X) = Var(\mu+\sigma Z) = \sigma^2 Var(Z) = \sigma^2$$

(b) **Binomial**(n,π)**;**
Binomial p.m.f is given by

$$f_X(x) = \frac{n!}{x!(n-x)!}\pi^x(1-\pi)^{n-x}$$

Therefore,

$$E(X) = \sum_{x=0}^{n} x f_X(x) = \sum_{x=0}^{n} x\frac{n!}{x!(n-x)!}\pi^x(1-\pi)^{n-x}$$

The first term in the above summation will be zero so we could start it from 1. Also cancelling the common factors of x in numerator and denominator.

$$E(X) = \sum_{x=1}^{n}\frac{n!}{(x-1)!(n-x)!}\pi^x(1-\pi)^{n-x}$$

Making change of variable $x' = x - 1$ above we get,

$$E(X) = \sum_{x'=0}^{n-1}\frac{n!}{x'!(n-x'-1)!}\pi^{x'+1}(1-\pi)^{n-x'-1}$$

$$E(X) = n\pi \sum_{x'=0}^{n-1} \frac{(n-1)!}{x'!(n-x'-1)!} \pi^{x'} (1-\pi)^{n-x'-1}$$

The terms in the summation are just the binomial funciton for $n-1$ trials, and we are summing it over all values of x so sum is 1.

$$E(X) = n\pi$$

Now,

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=0}^{n} x^2 \frac{n!}{x!(n-x)!} \pi^x (1-\pi)^{n-x} = n\pi \sum_{x=0}^{n-1} (x+1) \frac{(n-1)!}{x!(n-x-1)!} \pi^x (1-\pi)^{n-1-x}$$

$$E(X^2) = n\pi \underbrace{\sum_{x=0}^{n-1} \frac{(n-1)!}{x!(n-x-1)!} \pi^x (1-\pi)^{n-1-x}}_{1} + n\pi \underbrace{\sum_{x=0}^{n-1} x \frac{(n-1)!}{x!(n-x-1)!} \pi^x (1-\pi)^{n-1-x}}_{(n-1)\pi}$$

$$E(X^2) = n\pi + n(n-1)\pi^2$$

Therefore,

$$Var(X) = n\pi(1-\pi)$$

(c) **Poisson** (λ)**;**
Poisson : $p(x, \lambda) = \frac{\lambda^x}{x!} e^{-\lambda}$

$$E(X) = \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} = \lambda \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x}{x!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda} \sum_{x=0}^{\infty} (x+1) \frac{\lambda^x}{x!}$$

$$E(X^2) = \lambda e^{-\lambda} \left(\sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \right) = \lambda e^{-\lambda} (\lambda e^{\lambda} + e^{\lambda}) = \lambda^2 + \lambda$$

So,

$$Var(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

(d) **Exponential(λ);**
Exponential : $f_X(x) = \lambda e^{-\lambda x} \qquad x \geq 0.$

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x}dx = -xe^{-\lambda x} - \frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty = \lim_{x\to\infty}\left(-xe^{-\lambda x} - \frac{1}{\lambda}e^{-\lambda x}\right) + \frac{1}{\lambda} = \frac{1}{\lambda}$$

$$E(X^2) = \int_0^\infty x^2\lambda e^{-\lambda x}dx = -x^2e^{-\lambda x} - \frac{2}{\lambda}xe^{-\lambda x} - \frac{2}{\lambda^2}e^{-\lambda x}\Big|_0^\infty = \frac{2}{\lambda^2}$$

So,

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

4. **Suuppose that X and Y are jointly continuous. Show that $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy \qquad x$ $\Re$**

$$\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(a,b)dadb = F_{XY}(x,y) = P(X \leq x, Y \leq y)$$

Therefore,

$$\lim_{y\to\infty}\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(a,b)dadb = \lim_{y\to\infty} F_{XY}(x,y) = \lim_{y\to\infty} P(X \leq x, Y \leq y) = P(X \leq x) = F_X(x)$$

Now,

$$\frac{d}{dx}\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(a,b)dadb = \frac{d}{dx}F_X(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} f_{XY}(x,b)db = f_X(x)$$

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

5. **Suppose that X and Y are jointly Gaussian with parameters $\mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho$. Show that $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$.**
In this case we have,

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_y\sigma_x\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy$$

$$\vdots$$

(Hint : Do substitution of variables and Complete the squares)

6. **Suppose $X \sim \mathcal{N}(0,1)$, and define $Y = X^2$. Are X and Y uncorreleated? Are X and Y independent? Find the pdf of Y. Are X and Y jointly continuous?**

Note that $E[X] = 0$ and $E[Y] = E[X^2] = \sigma_x^2 = 1$. Then

$$\rho = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{E[(X-\mu_x)(Y-\mu_y)]}{\sigma_x\sigma_y} = \frac{E[X(X^2-1)]}{\sigma_x\sigma_y} = \frac{E[X^3]-E[X]}{\sigma_x\sigma_y}$$

But for a Gaussian with mean zero, all odd moments are 0, so $E[X^3] = 0$. So $\rho = 0$, and X and Y are uncorrelated. As $Y = X^2$, Y cannot be independent of X — they are functionally related.

$$\{Y \leq y\} = \{X^2 \leq y\} = \{-\sqrt{y} \leq X \leq \sqrt{y}\} = \{-\sqrt{y} < X \leq \sqrt{y}\} \cup \{X = -\sqrt{y}\}$$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) + P[X = -\sqrt{y}]$$

Now, X is a continuous r.v. so $P[X = -\sqrt{y}] = 0$, then for $y > 0$

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{1}{2\sqrt{y}}f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}}f_X(-\sqrt{y}) = \frac{1}{2\sqrt{y}\sqrt{2\pi}}\exp(-y/2) + \frac{1}{2\sqrt{y}\sqrt{2\pi}}\exp(-y/2)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}}e^{-\frac{1}{2}y}u(y)$$

where, $u(y)$ is the standard unit step function.

X and Y are jointly continuous: Look at the joint CDF:

$$\begin{aligned} F_{XY}(\alpha,\beta) &= P(X \leq \alpha, Y \leq \beta) = P(X \leq \alpha, X^2 \leq \beta) \\ &= P(X \leq \alpha, -\sqrt{\beta} \leq X \leq \sqrt{\beta}) \\ &= P(-\sqrt{\beta} \leq X, X \leq \min(\alpha, \sqrt{\beta})) \end{aligned}$$

This is a continuous function of α and β (as can be realized with a little thought).