Utah State University
ECE 6010
Stochastic Processes
Homework # 1 Solutions

1. Create a list of all the stochastic processes you can think of that might occur in the real world (not just examples from the textbook). Be creative!

2. We defined a field to be a collection of sets that is closed under complementation and finite unions. Show that such a collection is also closed under finite intersections.

   Given the field $\mathcal{F}$ we know that for $A, B \in \mathcal{F}$, we have $A \cup B \in \mathcal{F}$ and $A^c, B^c \in \mathcal{F}$. Generalizing, when $A_i \in \mathcal{F}, i = 1, \ldots, n$, then $\cup_{i=1}^n A_i \in \mathcal{F}$. Furthermore, $\cup_{i=1}^n A_i^c \in \mathcal{F}$, as must be the complementary event,

   $$[\cup_{i=1}^n A_i^c]^c$$

   Hence, by DeMorgan's theorem,

   $$\cap_{i=1}^n A_i \in \mathcal{F}.$$

3. Using the axioms of probability, prove the following properties of probability:

   (a) $P(A^c) = 1 - P(A)$
   $A^c \cap A = \emptyset$ and $A^c \cup A = \Omega$.
   $P(A^c \cup A) = P(\Omega) = 1$ but also $P(A^c \cup A) = P(A^c) + P(A)$ (by additivity), so $1 = P(A^c) + P(A)$.

   (b) $P(\emptyset) = 0$
   $1 = P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset)$.

   (c) $A \subset B \Rightarrow P(A) \leq P(B)$
   If $A \subset B$ then $B = A \cup (B \setminus A)$. So

   $$P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A) \geq P(A).$$

   (d) $P(A \cup B) = P(A) + P(B) - P(AB)$

   $$P(A \cup B) = P(AB^c \cup A^c B \cup AB) = P(AB^c) + P(A^c B) + P(AB). \qquad (*)$$

   Now

   $$A = AB \cup AB^c \qquad B = AB \cup A^c B$$

so

$$P(A) = P(AB) + P(AB^c) \qquad\qquad P(B) = P(AB) + P(A^cB)$$

and so

$$P(AB^c) = P(A) - P(AB) \qquad\qquad P(A^cB) = P(B)P(AB).$$

Substituting these in (**) gives the answer.

(e) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$
This is true for $n = 2$ events: $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$. Proof by induction: Assume true for n:

$$P(A_1 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} P(A_i),$$

and show that it holds for $n + 1$:

$$P(A_1 \cup A_2 \cup \cdots \cup A_{n+1}) = P([A_1 \cup A_2 \cup \cdots \cup A_n] \cup A_{n+1})$$

But the set $[A_1 \cup A_2 \cup \cdots \cup A_n]$ describes a single set, which we will call B. We then have $P(B \cup A_{n+1}) \leq P(A) + P(A_{n+1})$. But by the inductive hypothesis, we have a bound on $P(B)$:

$$P(B) = P(A_1 \cup \cdots \cup A_n) \leq \sum_{i=1}^{n} P(A_i),$$

so that

$$P(A_1 \cup \cdots \cup A_n \cup A_{n+1}) \leq \sum_{i=1}^{n} P(A_i) + P(A_{n+1})$$

4. Suppose $P(B) > 0$. Prove the following properties of conditional probability:

   (a) $P(A|B) \geq 0$.
   $P(A|B) = P(AB)/P(B)$. But since $P(AB) \geq 0$, the result follows.

   (b) $P(\Omega|B) = 1$
   $P(\Omega|B) = P(\Omega B)/P(B) = P(B)/P(B) = 1$.

   (c) For $A_1, A_2, \ldots \in \mathcal{F}$ with $A_iA_j = \emptyset$ for $i \neq j$, $P(\cup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} P(A_i|B)$
   $P(\cup_{i=1}^{\infty} A_i|B) = P((\cup_{i=1}^{\infty} A_i)B)/P(B) = P(\cup_{i=1}^{\infty} A_iB)/P(B) = \sum_{i=1}^{\infty} P(A_iB)/P(B) = \sum_{i=1}^{\infty} P(A_i|B)$, since the A_iB sets are disjoint.

(d) $AB = \emptyset \Rightarrow P(A|B) = 0$.
$P(A|B) = P(AB)/P(B) = P(\emptyset)/P(B) = 0$.

(e) $P(B|B) = 1$
$P(B|B) = P(BB)/P(B) = P(B)/P(B) = 1$

(f) $A \subset B \Rightarrow P(A|B) \geq P(A)$
Since $A \subset B$, $AB = A$. Then

$$P(A|B) = P(AB)/P(B) = P(A)/P(B) \geq P(A)$$

since $P(B) \leq 1$.

(g) $B \subset A \Rightarrow P(A|B) = 1$.
Since $A \subset B$, $AB = B$. Then

$$P(A|B) = P(AB)/P(B) = P(B)/P(B) = 1.$$

5. Prove the law of total probability.

Let $\{A_i\}$ be a partition of A. Note that $A = \cup_{i=1}^{n} A_i A$. Note also that $P(AA_i) = P(A|A_i)P(A_i)$.

$$P(A) = P(\cup_{i=1}^{n} AA_i) = \sum_{i=1}^{n} P(AA_i) = \sum_{i=1}^{n} P(A|A_i)P(A_i).$$

6. Prove Bayes rule

$$P(A|B) = P(AB)/P(B) = P(B|A)P(A)/P(B)$$

7. Suppose A and B are independent events. Show that A and B^c are also independent.

By independence, $P(AB) = P(A)P(B)$.

$$P(A) = P(A(B \cup B^c)) = P(AB \cup AB^c) = P(AB) + P(AB^c) = P(A)P(B) + P(AB^c)$$

so

$$P(AB^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c).$$

which means A and B^c are independent.