Utah State University
ECE 6010
Stochastic Processes
Homework # 11 Solutions

1. Let M_n denote the sequence of sample means from an iid random process X_n:

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

(a) Is M_n a Markov process?

$$\begin{aligned} M_n &= \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}[X_n + (n-1)M_{n-1}] \\ &= \frac{1}{n}X_n + (1-\frac{1}{n})M_{n-1} \end{aligned}$$

Clearly if M_{n-1} is given then M_n depends only on X_n and is independent of $M_{n-2}, M_{n-3}, \ldots$. Therefore, M_n is a Markov process.

(b) If the answer to part a is yes, find the following state transition pdf: $f_{M_n}(x|M_{n-1} = y)$.

$$\begin{aligned} f_{M_n}(x|M_{n-1} = y) &= P[x < M_n \le x + dx \mid M_{n-1} = y] \\ &= P[x < \frac{1}{n} + (1-\frac{1}{n})y \le x + dx] \\ &= P[nx - (n-1)y < x_n \le nx - (n-1)y + dx] \\ &= f_x(nx - (n-1)y)dx \end{aligned}$$

2. An urn initially contains five black balls and five white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn; if the ball is white it is put back in the urn, otherwise it is left out. Let X_n be the number of black balls remaining in the urn after n draws from the urn.

(a) Is X_n a Markov process? If so, find the apropriate transition probabilities.

(b) Do the transition probabilities depend on n?

The number X_n of black balls in the urn completely specifies the probability of the outcomes of a trial; therefore X_n is independent of its past values and X_n is a Markov proces.

$$\begin{aligned} P[X_n = 4|X_{n-1} = 5] &= \frac{5}{10} = 1 - P[X_n = 5|X_{n-1} = 5] \\ P[X_n = 3|X_{n-1} = 4] &= \frac{4}{9} = 1 - P[X_n = 4|X_{n-1} = 4] \\ P[X_n = 2|X_{n-1} = 3] &= \frac{3}{8} = 1 - P[X_n = 3|X_{n-1} = 3] \\ P[X_n = 1|X_{n-1} = 2] &= \frac{2}{7} = 1 - P[X_n = 2|X_{n-1} = 2] \\ P[X_n = 0|X_{n-1} = 1] &= \frac{1}{6} = 1 - P[X_n = 1|X_{n-1} = 1] \\ P[X_n = 0|X_{n-1} = 0] &= 1 \end{aligned}$$

All the transition probability are independent of time.

3. Let X_n be the Bernoulli iid process, and let Y_n be given by $Y_n = X_n + X_{n-1}$. It was shown in Example 8.2 that y_n is not a Markov process. Consider the vector process defined by $\mathbf{Z}_n = (X_n, X_{n-1})$.

(a) Show that $\mathbf{Z}_n$ is a Markov process.

$$P[\mathbf{Z}_{n+1} = (x_{n+1}, x_n)| \underbrace{\mathbf{Z}_n = (x_n, x_{n-1}), \mathbf{Z}_{n-1} = (x_{n-1}, x_{n-2}), \ldots}_{\text{all past vectors}}]$$

$$\begin{aligned} &= P[\mathbf{Z}_{n+1} = (x_{n+1}, x_n)| \underbrace{X_n = x_n, X_{n-1} \ldots}_{\text{all past Bernoulli trials}}] \\ &= P[\underbrace{X_{n+1} = x_{n+1}}_{\text{next trial}}] \\ &= P[\mathbf{Z}_{n+1} = (x_{n+1}, x_n)|\mathbf{Z}_n = (x_n, x_{n-1})] \end{aligned}$$

Therefore, $\mathbf{Z}_n$ is a Markov process.

(b) Find the state transition diagram for $\mathbf{Z}_n$.

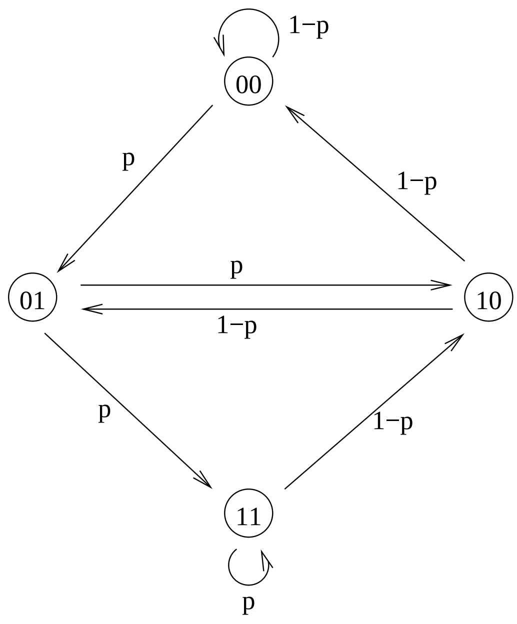


Figure 1: State transition diagram for $\mathbf{Z}_n$

where $p = P[x = 1]$.

4. Show that the following autoregressive process is a Markov process: $Y_n = rY_{n-1} + X_n$ where $Y_0 = 0$ and X_n is an iid process.

$$Y_n = rY_{n-1} + X_n \qquad Y_0 = 0 \qquad Y_n - rY_{n-1} = X_n$$

$$\begin{aligned} f_{Y_n}(y|Y_{n-1} = y_1 \; Y_{n-2} = y_{n-2} \; \ldots) &= f_{X_n}(y - ry_1) \\ &= f_{Y_n}(y|Y_{n-1} = y_1) \end{aligned}$$

$\Rightarrow Y_n$ is a Markov process.

5. Let X_n be the Markov chain defined in Problem 2.

(a) Find the one-step transition probability matrix P for X_n.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{7} & \frac{5}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & \frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{10} & \frac{5}{10} \end{bmatrix}$$

(b) Find the two-step transition probability matrix P^2 by matrix multiplication. Check your answer by computing $p_{54}(2)$ and comparing it to the correspoinding entry in P^2.

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{11}{36} & \frac{25}{36} & 0 & 0 & 0 & 0 \\ \frac{1}{21} & \frac{65}{144} & \frac{25}{49} & 0 & 0 & 0 \\ 0 & \frac{3}{28} & \frac{225}{448} & \frac{25}{64} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{95}{192} & \frac{25}{81} & 0 \\ 0 & 0 & 0 & \frac{2}{9} & \frac{19}{36} & \frac{1}{4} \end{bmatrix}$$

$$p_{54}(2) = p_{55}^{no\ change}(1)p_{54}^{no\ change}(1) + p_{54}^{no\ change}(1)p_{53}^{no\ change}(1) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{9} = \frac{19}{36}$$

From the P^2 matrix $p_{54}(2) = \frac{19}{36}$.

(c) What happens to X_n as n approaches infinity ? Use your answer to guess the limit of P^n as $n \to \infty$.
As $n \to \infty$ eventually all black balls are removed. Thus

$$P^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. Two gambler play the following game. A fair coin is flipped; if the outcome is heads, player A pays player B \$ 1, and if the outcoime is tails player B plays player A \$ 1. the game is continued until one of the players goes broke. Suppose that initially player A has \$ 1 and player B has \$ 2, so a total of \$ 3 is up for grabs. Let X_n denote the number of dolars held by player A after n trials.

(a) Show that X_n is a Markov chain.
$X_n \in 0, 1, 2, 3$
$P[X_n = k | X_{n-1} = j, \ldots] = P[X_n = k | X_{n-1} = j]$
since $X_n = X_{n-1} \pm 1$ for $X_{n-1} \in \{1, 2\}$
and $X_n = X_{n-1}$ if $X_{n-1} \in \{0, 3\}$.

(b) Sketch the state transition diagram for X_n and give the one-step transition probability matrix P.

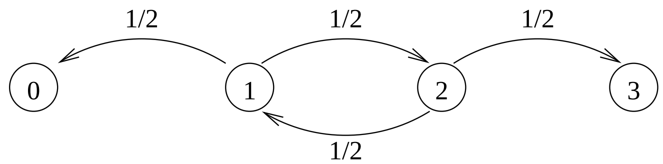


Figure 2: State transition diagram for X_n.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Use the state trasition diagram to help you show that for n even $p_{ii}(n) = (1/2)^n$ for $i = 1, 2$ and $p_{10}(n) = 2/3(1 - (1/4)^k) = p_{23}(n)$.

For $n = 2k,\ i \in \{1, 2\}$

$$\begin{aligned}
p_{ii}(n) &= P[\overbrace{HT\ HT\ HT\ \ldots\ HT}^{2k}] = \left(\frac{1}{2}\right)^n \\
p_{10}(n) &= \sum_{j=0}^{k-1} P[j\ 1 \rightarrow\ 2 \text{ cycles and then go to } 0] \\
&= \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^{2j} \frac{1}{2} \\
&= \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^k\right) \\
p_{23}(n) &= p_{10}(n) \text{ by symmetry.}
\end{aligned}$$

(d) Find the n-step transition probability matrix for n even using part c.

$$P(n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^k\right) & \left(\frac{1}{2}\right)^n & 0 & \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^k\right) \\ \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^k\right) & 0 & \left(\frac{1}{2}\right)^n & \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^k\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Find the limit of P^n as $n \rightarrow \infty$.

$$P(n) \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(f) Find the probability that player A eventually wins.

$$\begin{aligned}
\mathbf{p}(n) &= [0\ 1\ 0\ 0]P(n) \\
&= \left[\frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^k\right), \left(\frac{1}{4}\right)^k, 0, \frac{1}{3}\left(1 - \left(\frac{1}{4}\right)^k\right)\right] \\
&\rightarrow \left[\frac{2}{3},\ 0,\ 0,\ \frac{1}{3}\right]
\end{aligned}$$

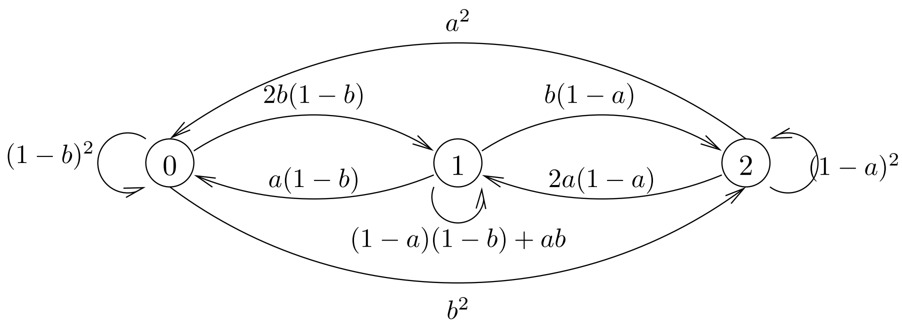
$P[\text{player } A \text{ wins}] = \frac{1}{3}$.

7. A machine consists of two parts that fail and are repaired independently. A working part fails during any given day with probability a. A part that is not working is repaired by the next day with probability b. Let X_n be the number of working parts in day n.

(a) Show that x_n is a three-state Markov chain and give its one-step transition probability matrix P.

$X_n \in \{0, 1, 2\}$

$$P = \begin{bmatrix} (1-b)^2 & 2b(1-b) & b^2 \\ a(1-b) & (1-a)(1-b) + ab & b(1-b) \\ a^2 & 2a(1-a) & (1-a)^2 \end{bmatrix}$$



(b) Show that the steady state pmf $\underline{\pi}$ is binomial with parameter $p = b/(a+b)$.

Claim: the steady state pmf is

$$\begin{aligned}
\underline{\pi} &= \left[\left(\frac{a}{a+b}\right)^2, 2\left(\frac{b}{a+b}\right)\left(\frac{a}{a+b}\right), \left(\frac{b}{a+b}\right)^2\right] \\
&= \frac{1}{(a+b)^2}(a^2, 2ab, b^2)
\end{aligned}$$

$$\begin{aligned}
\underline{\pi}P &= \frac{1}{(a+b)^2}(a^2, 2ab, b^2)\begin{bmatrix} (1-b)^2 & 2b(1-b) & b^2 \\ a(1-b) & (1-a)(1-b)+ab & b(1-a) \\ a^2 & 2a(1-a) & (1-a)^2 \end{bmatrix} \\
&= \frac{1}{(a+b)^2}\begin{bmatrix} a^2(1-b)^2 + 2a^2b(1-b) + a^2b^2 \\ 2a^2b(1-b) + 2ab(1-a)(1-b) + 2a^2b^2 + 2ab^2(1-a) \\ a^2b^2 + 2ab^2(1-a) + b^2(1-a)^2 \end{bmatrix} \\
&= \frac{1}{(a+b)^2}(a^2, 2ab, b^2) \\
&= \underline{\pi}
\end{aligned}$$

(c) What do you expect is steady state pmf for a machine that consists of n parts?

$$P[X_n = k] = \binom{n}{k}\left(\frac{a}{a+b}\right)^k\left(\frac{b}{a+b}\right)^{n-k} \qquad 0 \le k \le n$$