ECE 3640
Lecture 9 – The Discrete-time Fourier Transform

Objective: To learn about the discrete-time Fourier transform (DTFT), which provides a spectral representation for discrete-time signals.

Transforms we have met and loved

We have studied this year a variety of transforms:

Laplace transforms which are useful for system analysis, including transient and stability analysis. By evaluating at $s = j\omega$ we explored also the concept of frequency response.

Z-transforms which are the transform appropriate for discrete-time systems. Like the Laplace transform, it can be used for transient analysis, stability analysis, and, by evaluating at $z = e^{j\omega T}$ we get the concept of frequency response.

Fourier series which are used to provide a representation of periodic signals. This has some application to circuit analysis for periodic signals, and leads, by taking a limit to signals of increasingly longer period, to the Fourier transform. We also saw that we can take the idea of series representations of functions and use a variety of other basis functions for other useful representations.

Fourier transforms which can be used to examine frequency response of signals. By means of their properties, we are also lead to consider concepts such as modulation. Fourier transforms do not really address the stability issues that Laplace transforms do, nor can they be used as conveniently for transient analysis. However, by not starting at $t = 0$, they simplify some other issues.

Two more transforms are introduced:

The Discrete-time Fourier Transform is to the Z-transform what the Fourier transform is to the Laplace transform. That is, we have an exact frequency component representation of signals that are **not periodic** by evaluating a (possibly two-sided) Z-transform at $z = e^{j\Omega}$. The DTFT is the study of this set of lecture notes.

The Discrete Fourier Transform (DFT) can be used to compute a transform of a finite-length discretely-sampled set of data. The DFT can be used for computational signal analysis, and its implementation in the form of the FFT is very common. However, because the signal is truncated in time and in frequency, it does not provide an exact frequency analysis (although there are techniques to get close enough in practice).

The Discrete-time Fourier Transform (DTFT)

Recall that the Z-transform is

$$F(z) = \sum_{n=0}^{\infty} f[n] z^{-n}.$$

The DTFT is defined as

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\Omega n}$$

Clearly, if $f[n]$ is causal, the DTFT is the Z-transform evaluated on the unit circle. (Just like the FT is the Laplace transform evaluated on the $j\omega$ axis.) Observe that $F(\Omega)$ is 2π-periodic:

$$F(\Omega) = F(\Omega + 2\pi).$$

Again we see this idea of the periodic repetition of the spectrum in the frequency domain and the source of aliasing.

Note that $F(\Omega)$ is a function of continuous frequency Ω — we are not looking at samples of the spectrum (as for the DFT), and hence the function does not have to be periodic.

But, $F(\Omega)$ is periodic, and hence, $F(\Omega)$ has a Fourier series representation: its F.S. coefficient are just the samples $f[k]$. But this representation allows us to write down an **inverse DTFT**: to go from $F(\Omega)$ to $f[k]$, simply stick things in the formula for the F.S. coefficients, where the period of the function is 2π

$$f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{jk\Omega} d\Omega.$$

(Need we point out that, yet again, in going from time domain to frequency domain the exponent has negative sign, and in going from frequency domain to time domain the exponent has a positive sign?) Notationally we will write

$$f[k] \Leftrightarrow F(\Omega).$$

And, as for other transforms, we can talk about the amplitude and phase spectra.

Example 1 Find the DTFT of $f[k] = a^k u[k]$.

$$F(\Omega) = \sum_{k=0}^{\infty} a^k e^{-j\Omega k} = \frac{1}{1 - ae^{-j\Omega}}.$$

(No surprise there!). Magnitude response:

$$|F(\Omega)| = \frac{1}{\sqrt{(1 - a\cos\Omega)^2 + (a\sin\Omega)^2}}$$

Phase response:

$$\angle F(\Omega) = -\tan^{-1}\left[\frac{a\sin\Omega}{1 - a\cos\Omega}\right]$$

Show plot. Interpret the frequency axis: The frequency $\Omega = \pi$ represents half the sampling rate. Any higher frequencies get wrapped back around. □

Example 2 An example that leads to some important insight when dealing with the DFT is the discrete-time "rect" function.

$$f[k] = \begin{cases} 1 & |k| \le N \\ 0 & \text{otherwise} \end{cases}$$

Then

$$F(\Omega) = \sum_{k=-N}^{N} e^{-jk\Omega} = \frac{e^{-j\Omega(N+1)} - e^{j\Omega N}}{e^{j\Omega} - 1}.$$

(See summation formula on p. 64.) Then doing the old trick of pulling out enough exponent to produce real trigonometric functions we get

$$F(\Omega) = \frac{e^{-j\Omega/2}(e^{-j\Omega(2N+1)/2} - e^{j\Omega(2N+1)/2})}{e^{-j\Omega/2}(e^{-j\Omega/2} - e^{j\Omega/2})} = \frac{\sin((2N+1)/2\Omega)}{\sin(.5\Omega)}$$

Plot this: when does it cross the axis. This is kind of like a sinc function, except that it is periodic. □

(Comment on spectral leakage).

Example 3 We can also compute an inverse transform: Let

$$F(\Omega) = \mathrm{rect}(\Omega/(\pi/2))$$

and its periodic repetition. Then

$$f[k] = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{jk\Omega} d\Omega = \frac{1}{4}\,\mathrm{sinc}(k\pi/4).$$

□

System analysis using the DTFT

For a discrete-time system with input signal $f[k]$ and impulse response $h[k]$, the output is

$$y[k] = f[k] * h[k].$$

The discrete-time convolution theorem says that we can transform this to get

$$Y(\Omega) = F(\Omega)H(\Omega).$$

Same sort of stuff we have seen all year long. We will finish the story by an example:

Example 4 Let $h[k] = (0.5)^k u[k]$ and $f[k] = (0.8)^k u[k]$. Find the output.

We know that we could convolve this, but we will instead (to illustrate the point) use the convolution theorem. We can find (from our previous example, if nothing else, that

$$F(\Omega) = \frac{1}{1 - .8e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - .8}$$

$$H(\Omega) = \frac{1}{1 - .5e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - .5}$$

We find

$$Y(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - .8}\frac{e^{j\Omega}}{e^{j\Omega} - .5}$$

Taking the inverse transform (by mean of PFE, just like we would have done for the Z-transform) we get

$$y(t) = [-\frac{5}{3}(0.5)^k + \frac{8}{3}(0.8)^k]u[k].$$

□

Properties of the DTFT

1. $f[-k] \leftrightarrow F(-\Omega)$
2. $kf[k] \leftrightarrow j\frac{dF(\Omega)}{d\Omega}$
3. $f[k-k_0] \leftrightarrow F(\Omega)e^{-jk_0\Omega}$
4. $f[k]e^{jk\Omega_s} \leftrightarrow F(\Omega - \Omega_s)$
5. $f_1[k]f_2[k] \leftrightarrow \frac{1}{2\pi}F_1(\Omega) * F_2(\Omega)$.
6. Parseval:
$$E_f = \sum_k |f[k]|^2 = \frac{1}{2\pi}\int_{2\pi} |F(\Omega)|^2 \, d\Omega.$$