ECE 3640
Lecture 6 – Sampling

Objective: To learn and prove the sampling theorem and understand its implications.

The sampling theorem

Due to the increased use of computers in all engineering applications, including signal processing, it is important to spend some more time examining issues of sampling. In this chapter we will look at sampling both in the time domain and the frequency domain.

We have already encountered the sampling theorem and, arguing purely from a trigonometric-identity point of view, have established the Nyquist sampling criterion for sinusoidal signals. However, we have not fully addressed the sampling of more general signals, nor provided a general proof. Nor have we indicated how to reconstruct a signal from its samples. With the tools of Fourier transforms and Fourier series available to us we are now ready to finish the job that was started months ago.

To begin with, suppose we have a signal $x(t)$ which we wish to sample. Let us suppose further that the signal is bandlimited to B Hz. This means that its Fourier transform is nonzero for $-2\pi B < \omega < 2\pi B$. Plot spectrum.

We will model the sampling process as multiplication of $x(t)$ by the "picket fence" function

$$\delta_T(t) = \sum_n \delta(t - nT).$$

We encountered this periodic function when we studied Fourier series. Recall that by its Fourier series representation we can write

$$\delta_T(t) = \frac{1}{T} \sum_n e^{jn\omega_s t}$$

where $\omega_s = \frac{2\pi}{T}$. The frequency $f_s = \omega_s/(2\pi) = 1/T$ is the sampling frequency in samples/sec. Suppose that the sampling frequency is chosen so that $f_s > 2B$, or equivalently, $\omega_s > 4\pi B$.

The sampled output is denoted as $\overline{x}(t)$, where

$$\overline{x}(t) = x(t)\delta_T(t)$$

Using the F.S. representation we get

$$\overline{x}(t) = x(t)\frac{1}{T} \sum_n e^{jn\omega_s t}$$

Now lets look at the spectrum of the transformed signal. Using the convolution property,

$$\overline{X}(\omega) = \frac{1}{2\pi}\frac{1}{T} X(\omega) * \sum_n 2\pi\delta(\omega - n\omega_s) = \frac{1}{T} \sum_n X(\omega - n\omega_s).$$

Plot the spectrum of the sampled signal with both ω frequency and f frequency. Observe the following:

- The spectrum is periodic, with period 2π, because of the multiple copies of the spectrum.
- The spectrum is scaled down by a factor of $1/T$.
- Note that *in this case* there is no overlap between the images of the spectrum.

Now consider the effect of reducing the sampling rate to $f_s < 2B$. In this case, the duplicates of the spectrum overlap each other. The overlap of the spectrum is **aliasing**.

This demonstration more-or-less proves the sampling theorem for general signals. Provided that we sample fast enough, the signal spectrum is not distorted by the sampling process. If we don't sample fast enough, there will be distortion. The next question is: given a set of samples, how do we get the signal back? From the spectrum, the answer is to filter the signal with a lowpass filter with cutoff $\omega_c \geq 2\pi B$. This cuts out the images and leaves us with the original spectrum. This is a sort of idealized point of view, because it assumes that we are filtering a continuous-time function $\overline{x}(t)$, which is a sequence of weighted delta functions. In practice, we have numbers $x[n]$ representing the value of the function $x[n] = x(nT) = x(n/f_s)$. How can we recover the time function from this?

Theorem 1 (The sampling theorem)
If $x(t)$ is bandlimited to B Hz then it can be recovered from signals taken at a sampling rate $f_s > 2B$. The recovery formula is

$$x(t) = \sum_n x(nT)g(t - nT)$$

where

$$g(t) = \frac{\sin(\pi f_s t)}{\pi f_s t} = \text{sinc}(\pi f_s t).$$

Show what the formula means: we are interpolating in time between samples using the sinc function.

We will prove this theorem. Because we are actually lacking a few theoretical tools, it will take a bit of work. What makes this interesting is we will end up using in a very essential way most of the transform ideas we have talked about.

1. The first step is to notice that the spectrum of the sampled signal,

   $$\overline{X}(\omega) = \frac{1}{T}\sum_n X(\omega - n\omega_s)$$

   is periodic and hence has a Fourier series. The period of the function in frequency is ω_s, and the fundamental frequency is

   $$p_0 = \frac{2\pi}{\omega_s} = \frac{1}{f_s} = T.$$

   By the F.S. we can write

   $$\overline{X}(\omega) = \sum_n c_n e^{jn\omega T}$$

   where the c_n are the F.S. coefficients

   $$c_n = \frac{1}{\omega_s}\int_{\omega_s} \overline{X}(\omega)e^{-jn\omega T}d\omega = \frac{2\pi}{\omega_s}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{T}X(\omega)e^{-jn\omega T}d\omega.$$

But the integral is just the inverse F.T., evaluated at $t = -nT$:

$$c_n = \frac{1}{T}\left.\frac{2\pi}{\omega_s}x(t)\right|_{t=-nT} = x(-nT),$$

so

$$\overline{X}(\omega) = \sum_n x(-nt)e^{jn\omega T} = \sum_n x(nt)e^{-jn\omega T}.$$

2. Let $g(t) = \text{sinc}(\pi f_s t)$. Then

$$g(t) \Leftrightarrow T\,\text{rect}\left(\frac{\omega}{2\pi f_s}\right).$$

3. Let

$$y(t) = \sum_n x(nT)g(t - nT).$$

We will show that $y(t) = x(t)$ by showing that $Y(\omega) = X(\omega)$. We can compute the F.T. of $y(t)$ using linearity and the shifting property:

$$Y(\omega) = \sum_n x(nT)T\,\text{rect}\left(\frac{\omega}{2\pi f_s}\right)e^{-j\omega nT} = T\,\text{rect}\left(\frac{\omega}{2\pi f_s}\right)\sum_n x(nT)e^{-j\omega nT}$$

Observe that the summation on the right is the same as the F.S. we derived in step 1:

$$Y(\omega) = T\,\text{rect}\,\frac{\omega}{2\pi f_s}\overline{X}(\omega).$$

Now substituting in the spectrum of the sampled signal (derived above)

$$Y(\omega) = T\,\text{rect}\left(\frac{\omega}{2\pi f_s}\right)\left(\frac{1}{T}\sum_n X(\omega - n\omega_s)\right) = X(\omega)$$

since $x(t)$ is bandlimited to $-\pi f_s < \omega < \pi f_s$ or $-f_s/2 < f < f_s/2$.

Notice that the reconstruction filter is based upon a sinc function, whose transform is a rect function: we are really just doing the filtering implied by our initial intuition.

In practice, of course, we want to sample at a frequency higher than just twice the bandwidth to allow room for filter rolloff.

Other issues for sampling

Antialias filtering When sampling is done, the signal is usually passed first through an analog low-pass filter with the cutoff frequency set to ensure that no aliasing occurs.

The discussion above assumes that the signal is sampled by a sequence of impulse functions. In practice, the signal is sampled by multiplying by some other function. Is it still possible to recover the signal exactly? Provided that we satisfy the Nyquist sampling criterion, the answer is yes. The reconstruction is similar.

Some applications

Why digital?

1. Recoverable signals

2. Flexibility
3. Channel coding theorem. Source coding theorem.
4. Encryption

- Time division multiplexing
- Pulse code modulation

Spectral sampling

Just as we can sample a band-limited signal in the time domain and reconstruct it, provided that we sample often enough, so can we also sample a time-limited signal in the frequency domain and exactly reconstruct the spectrum, provided that we take the samples close enough together in the spectrum. There is thus a dual to the sampling theorem which applies to sampling the spectrum. We can also use this kind of thinking to find the F.T. of a periodic signal.

Let $f(t)$ be a causal signal time-limited to τ seconds. Its F.T. is

$$F(\omega) = \int_0^\tau f(t)e^{-j\omega t}dt.$$

Now create a periodic extension of $f(t)$ by repeating it every T_0 seconds.

$$f_{T_0}(t) = \sum_n f(t - nT_0).$$

Since $f_{T_0}(t)$ is periodic, it has a F.S. and we can write

$$f_{T_0}(t) = \sum_n D_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/T_0$ and (when $\tau < T_0$)

$$D_n = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-jn\omega_0 t} = \frac{1}{T_0}\int_0^{\tau} f(t)e^{-jn\omega_0 t}.$$

Comparing the F.S. coefficient with the F.T. above, it follows that

$$D_n = \frac{1}{T_0}F(n\omega_0).$$

An implication of this is that we can find the F.S. coefficients by first taking the F.T. of one period of our signal, then sampling and scaling the F.T.

In terms of reconstructing the signal spectrum from its samples, we can see that as long as $T_0 > \tau$, the cycles of $f(t)$ do not overlap. We can then (at least in concept) reconstruct the entire spectrum of $F(\omega)$ from its samples. Conceptually, we time-limit the function, then take its F.T. The sampling condition can be expressed as follows:

$$T_0 > \tau$$

$$F_0 < \frac{1}{\tau}$$

So we can reconstruct from samples of the spectrum, provided that the samples are close enough by comparison with the time-limit of the function.