ECE 3640

Lecture 4 – Fourier series: expansions of periodic functions.

Objective: To build upon the ideas from the previous lecture to learn about Fourier series, which are series representations of **periodic** functions.

Periodic signals and representations

From the last lecture we learned how functions can be represented as a series of other functions:

$$f(t) = \sum_{k=1}^{n} c_k i_k(t).$$

We discussed how certain classes of things can be built using certain kinds of basis functions. In this lecture we will consider specifically functions that are periodic, and basic functions which are trigonometric. Then the series is said to be a Fourier series.

A signal $f(t)$ is said to be periodic with period T_0 if

$$f(t) = f(t + T_0)$$

for all t. Diagram on board. Note that this must be an everlasting signal. Also note that, if we know one period of the signal we can find the rest of it by **periodic extension**. The integral over a single period of the function is denoted by

$$\int_{T_0} f(t)dt.$$

When integrating over one period of a periodic function, it *does not matter* when we start. Usually it is convenient to start at the beginning of a period.

The building block functions that can be used to build up periodic functions are themselves periodic: we will use the set of sinusoids. If the period of $f(t)$ is T_0, let $\omega_0 = 2\pi/T_0$. The frequency ω_0 is said to be the *fundamental frequency*; the fundamental frequency is related to the period of the function. Furthermore, let $F_0 = 1/T_0$. We will represent the function $f(t)$ using the set of sinusoids

$$\begin{aligned} i_0(t) &= \cos(0t) = 1 \\ i_1(t) &= \cos(\omega_0 t + \theta_1) \\ i_2(t) &= \cos(2\omega_0 t + \theta_2) \\ &\vdots \end{aligned}$$

Then,

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

The frequency $n\omega_0$ is said to be the nth harmonic of ω_0.

Note that for each basis function associated with $f(t)$ there are actually two parameters: the amplitude C_n and the phase θ_n. It will often turn out to be more useful to represent the function using both sines and cosines. Note that we can write

$$C_n \cos(n\omega_0 t + \theta_n) = C_n \cos(\theta_n) \cos(n\omega_0 t) - C_n \sin(\theta_n) \sin(n\omega_0 t).$$

Now let

$$a_n = C_n \cos\theta_n \qquad b_n = -C_n \sin\theta_n$$

Then

$$C_n \cos(n\omega_0 t + \theta_n) = a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

Then the series representation can be

$$\begin{aligned} f(t) &= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \end{aligned}$$

The first of these is the **compact trigonometric Fourier series**. The second is the **trigonometric Fourier series.**. To go from one to the other use

$$C_0 = a_0$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}(-b_n/a_n).$$

To complete the representation we must be able to compute the coefficients. But this is the same sort of thing we did before. If we can show that the set of functions $\{\cos(n\omega_0 t), \sin(n\omega_0 t)\}$ is in fact an orthogonal set, then we can use the same formulas we did before. Check:

$$\int_{T_0} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & n = m \neq 0 \end{cases}$$

So pairs of cosine functions are orthogonal. Similarly,

$$\int_{T_0} \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt = \begin{cases} 0 & n \neq m \\ \frac{T_0}{2} & n = m \neq 0 \end{cases}$$

So pairs of sin functions are orthogonal. Also,

$$\int_{T_0} \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0 \text{ for all } n \text{ and } m.$$

So the sines and cosines are orthogonal to each other. This means that we can use our formulas:

$$a_n = \frac{\langle f(t), \cos(n\omega_0 t)\rangle}{\langle \cos(n\omega_0 t), \cos(n\omega_0 t)\rangle} = \frac{2}{T_0}\int_{T_0} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{\langle f(t), \sin(n\omega_0 t)\rangle}{\langle \sin(n\omega_0 t), \sin(n\omega_0 t)\rangle} = \frac{2}{T_0}\int_{T_0} f(t)\sin(n\omega_0 t)\,dt$$

The only exception is the coefficient a_0:

$$a_0 = \frac{\langle f(t), 1\rangle}{\langle 1, 1\rangle} = \frac{1}{T_0}\int_{T_0} f(t)\,dt$$

Example 1

Let us start with a square wave:

$$f(t) = \begin{cases} 1 & -\pi/2 < t < \pi/2 \\ 0 & t \in [-\pi, -\pi/2), (\pi, 2, \pi] \end{cases}$$

and its periodic extension. The period is $T_0 = 2\pi$, so $\omega_0 = 2\pi/T_0 = 1$. The series is

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

The coefficients can be found as follows:

$$a_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dt = \frac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos nt\, dt = \frac{2}{n\pi} \sin(n\pi/2) = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n = 1, 5, 9, 13, \ldots \\ -\frac{2}{\pi n} & n = 3, 7, 11, 15, \ldots \end{cases}$$

$$b_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin nt\, dt = 0.$$

We can write

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \left(\cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \frac{1}{7} \cos 7t + \cdots \right).$$

Here is an interesting tidbit: Evaluate this at $t = 0$:

$$1 = \frac{1}{2} + \frac{2}{\pi} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right).$$

Solving, we get

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right).$$

In the compact form,

$$C_0 = \frac{1}{2}$$

$$C_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n} & n \text{ odd} \end{cases}$$

$$\theta_n = \begin{cases} -\pi & n = 3, 7, 11, 15, \ldots, \\ 0 & \text{otherwise} \end{cases}$$

□

It is interesting to see how the function gets built up at the pieces are added together:

The Fourier Spectrum

For a function $f(t)$ having a compact trigonometric Fourier series, the set of amplitudes $\{C_n\}$ and the set of phases $\{\theta_n\}$ provide all the information necessary to represent the function. (This is interesting, if you think about it: a function which is defined at every point in a continuum can be represented with only a countable number of points.) A plot of the amplitudes $\{C_n\}$ vs. the ω is the **amplitude spectrum** of the signal. A plot of the phase $\{\theta_n\}$ vs. ω is the **phase spectrum** of the signal.

Example 2 Show the magnitude and phase spectrum of the square wave from the previous example.

Actually, if we allow the amplitude to show a shift of π by allowing a signed quantity, then the task is somewhat easier:

In this case the spectrum is said to be an **amplitude spectrum**, rather than a magnitude spectrum. The amplitude spectrum is convenient to work with whenever all the sine terms are zero, so all the information is conveyed in the $\{a_n\}$. □

Example 3
Find the Fourier series of the following signal, and plot its magnitude and phase spectrum.

The signal is periodic with $T_0 = 2$. The fundamental frequency is

$$\omega_0 = \frac{2\pi}{2} = \pi.$$

The function can be written analytically (for the purposes of integration) as

$$f(t) = \begin{cases} 2At & -\frac{1}{2} \le t \le \frac{1}{2} \\ 2A(1-t) & \frac{1}{2} \le t \le \frac{3}{2} \end{cases}$$

This gives is only one period of the function, but that is sufficient for our purposes. The Fourier series coefficients can be written as

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_{-1/2}^{3/2} f(t)\cos n\pi t dt \\ &= \int_{-1/2}^{1/2} 2At\cos n\pi t dt + \int_{1/2}^{3/2} 2A(1-t)\cos n\pi t dt \\ &= 0 \end{aligned}$$

(We will take this at face value for now. Soon we will learn some tricks that will help evaluate this sort of thing sometimes. But **notice the utility of being able to use a symbolic integration package.** You may want to go back and review what we did early last quarter with MATLAB: the symbolic integration exercises were written with this sort of integration mind!)

$$
\begin{aligned}
b_n &= \frac{2}{T_0}\int_{-1/2}^{3/2} f(t)\sin n\pi t dt \\
&= \int_{-1/2}^{1/2} 2At\sin n\pi t dt + \int_{1/2}^{3/2} 2A(1-t)\sin n\pi t dt \\
&= \frac{8A}{n^2\pi^2}\sin(n\pi/2)
\end{aligned}
$$

Actually, I checked this using my current favorite symbolic tool, `Mathematica`. This is the result:

```
In[6] := bn = Integrate[2 A t Sin[n Pi t],{t,-1/2,1/2}] +
           Integrate[2 A (1-t) Sin[n Pi t],{t,1/2,3/2}]

                         n Pi          n Pi
          A (n Pi Cos[----] - 2 Sin[----])
                          2             2
Out[6]= -(--------------------------------) +
                           2   2
                          n  Pi

                    n Pi          n Pi
     A (n Pi Cos[----] + 2 Sin[----])
                     2             2
>    -------------------------------- +
                      2   2
                     n  Pi

                        n Pi
        -(A n Pi Cos[----])
                         2             n Pi
     2 (------------------ + A Sin[----])
                  2                    2
>    ------------------------------------ +
                        2   2
                       n  Pi

                    3 n Pi          3 n Pi
     A (n Pi Cos[------] - 2 Sin[------])
                      2               2
>    ------------------------------------
                        2   2
                       n  Pi


In[7]:= Simplify[bn]

                          n Pi          n Pi       n Pi 2
        -4 A (n Pi Cos[----] - 2 Sin[----]) Sin[----]
                           2             2          2
Out[7]= ---------------------------------------------
                                  2   2
```

```
                         n  Pi
```

In problems of this sort, it is considered bad form to leave it in terms of all the sins and cosines (go thy way and sin no more!), since they can occlude the underlying values of the coefficients. So we will work through the detail to get the answer shown in the book. In doing this, however, *do not* lose sight of the forest for the trees. We already have the coefficients; we are simply manipulating them a bit. In practice, these types of manipulations could be skipped or be done by a computer (say, if we were plotting the magnitude spectrum).

In the simplification, note that the term $\sin^2(n\pi/2)$ is zero when n is even. So we only need to consider odd n. For odd n, $\sin^2(n\pi/2) = 1$, so we can focus on the other part. First, consider odd n of the form $n = 4k + 1$. Then

$$\cos(n\pi/2) = \cos((4k + 1)\pi/2) = \cos(\pi/2) = 0$$

and

$$\sin(n\pi/2) = \sin((4k + 1)\pi/2) = \sin(\pi/2) = 1$$

$$\sin(3n\pi/2) = \sin(3(4k + 1)\pi/2) = \sin(3\pi/2) = -1$$

When $n = 4k + 3$,

$$\cos(n\pi/2) = \cos((4k + 3)\pi/2) = \cos(3\pi/2) = 0$$

$$\sin(n\pi/2) = \sin((4k + 3)\pi/2) = \sin(3\pi/2) = -1$$

$$\sin(3n\pi/2) = \sin(3(4k + 3)\pi/2) = \sin(9\pi/2) = 1$$

Combining all this together gives the desired answer.

Using the symbolic toolbox in MATLAB, I obtained

```
syms A n t
int('2*A*t*sin(n*pi*t)',t,-1/2,1/2)

ans =

-2*A*(-2*sin(1/2*n*pi)+n*pi*cos(1/2*n*pi))/n^2/pi^2
```

The Fourier series is

$$\begin{aligned} f(t) &= \sum_n b_n \sin(n\pi t) \\ &= \frac{8A}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin(n\pi t) \\ &= \frac{8A}{\pi^2} \sum_{n=1,3,5,\ldots} \frac{\sin(n\pi/2)}{n^2} \sin(n\pi t) \\ &= \frac{8A}{\pi^2} \left[\sin(\pi t) - \frac{1}{9}\sin(3\pi t) + \frac{1}{25}\sin(5\pi t) - \frac{1}{49}\sin(7\pi t) + \cdots \right] \end{aligned}$$

To plot the Fourier spectrum the compact trigonometric Fourier series is needed. Recall that this expresses the function in terms of cosines. We only have it in terms of sines. We we have to do a little phase-shift trick:

$$\sin(kt) = \cos(kt - 90^\circ)$$

$$-\sin(kt) = \cos(kt + 90^\circ)$$

Then the compact Fourier series is

$$f(t) = \frac{8A}{\pi^2}\left[\cos(\pi t - 90^\circ) + \frac{1}{9}\cos(3\pi t + 90^\circ) + \frac{1}{25}\cos(5\pi t - 90^\circ) + \frac{1}{49}\cos(7\pi t + 90^\circ) + \cdots\right].$$

The magnitude and phase spectrum are:

□

Symmetry and its effects

An **even** function $f(t)$ is a function such that

$$f(t) = f(-t)$$

Examples of even functions are $f(t) = \cos(t)$ and $f(t) = t^2$. An **odd** function is a function such that

$$f(t) = -f(-t)$$

Examples of odd functions are $f(t) = \sin(t)$ and $f(t) = t^3$. There are several facts about even and odd functions that can help us simplify and interpret some computations.

1. The product rules:
   - even × even = even
   - even × odd = odd × even = odd
   - odd × odd = even.

   For example, $t^2\cos(t)$ is an even function. $t\cos(t)$ is an odd function. (The rules are the same as the rules for adding even and odd numbers.)

2. Integration. When integrating over a symmetric interval about the origin,

$$\int_{-T_0/2}^{T_0/2} \text{even}(t)dt = 2\int_0^{T_0/2} \text{even}(t)dt$$

$$\int_{-T_0/2}^{T_0/2} \text{odd}(t)dt = 0$$

Let us use these facts in relation to Fourier series. Suppose we want to compute the F.S. of an **even** function (such as the square wave signal example). Then

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)dt = \frac{2}{T_0}\int_0^{T_0/2} f(t)dt.$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos(n\omega_0 t)dt = \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos(n\omega_0 t)dt.$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin(n\omega_0 t)dt = 0$$

To compute the F.S. of an **odd** signal,

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} f(t)dt = 0$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos(n\omega_0 t)dt = 0$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin(n\omega_0 t)dt = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin(n\omega_0 t)dt.$$

Review the signals transformed so far in light of these symmetries.

Determining the fundamental frequency

The trigonometric Fourier series can be used to represent any periodic function. In periodic functions, every frequency in the Fourier series representation is an integral multiple of some fundamental frequency. Such frequencies are said to be **harmonically related**. The ratio of any two harmonically related frequencies is a **rational number** (i.e., a number which can be represented as the ratio of two integers). (Interesting mathematical fact: there are more irrational numbers than there are rational numbers.) Any number which involves a transcendental number such as π or e, or which involves square roots which cannot be simplified down to ratios of integers (such as $\sqrt{2}$) is an irrational number. For functions which are harmonically related, the fundamental frequency is the greatest common divisor of the frequencies.

Example 4 Is the function $f_1(t) = 2 + 7\cos(\frac{1}{2}t + \theta_1) + \sqrt{3}\cos(\frac{2}{3}t + \theta_2) + 5\cos(\frac{7}{6}t)$ a periodic function? If it is, what is the fundamental frequency?

We need to determine if the frequencies are harmonically related. We neglect the DC term. Taking ratios:

$$\frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4}$$

which is rational.

$$\frac{\frac{1}{2}}{\frac{7}{6}} = \frac{3}{7}$$

which is rational.

$$\frac{\frac{2}{3}}{\frac{7}{6}} = \frac{4}{7}$$

which is rational. So $f(t)$ is a periodic function. The fundamental frequency is the greatest common divisor

$$\text{GCD}(1/2, 2/3, 7/6) = 1/6.$$

(The GCD(num)/LCM(den)). □

Example 5 Is the function $f(t) = 3\sin(3\sqrt{2}t + \theta) + 7\cos(6\sqrt{3}t)$ periodic?

The ratio of frequencies is

$$\frac{3\sqrt{2}}{6\sqrt{3}} = \frac{\sqrt{2}}{2\sqrt{3}}$$

which is irrational. □

Example 6 Is $f(t) = 3\sin(3\sqrt{2}t + \theta) + 7\cos(6\sqrt{2}t)$ periodic?

The ratio of frequencies is

$$\frac{3\sqrt{2}}{6\sqrt{2}}$$

which is rational. The fundamental frequency is

$$\text{GCD}(3\sqrt{2}, 6\sqrt{2}) = 3\sqrt{2}.$$

□

Interpretation of the smoothness of the function

Functions which are smooth (e.g. continuous) have most of their variations at lower frequencies. Functions which are not smooth have variations at higher frequencies. We can look at the rate of decay of the amplitude spectrum to determine something about the smoothness of the function.

For example, the square wave function has abrupt jumps and is not even continuous. The coefficients of the F.S. decay as $1/n$. By contrast, the sawtooth function we examined is smoother, since it is continuous. Its coefficients decay more quickly, decaying down as $1/n^2$.

The Gibbs phenomenon

According to our theory, with a complete set of basis functions we can represent any function exactly. We furthermore know how to obtain the best approximation if we use only a finite set of functions. Interestingly, even the best approximation can still have some substantial errors. Consider the error in the square-wave series. Observe that there is a jump just before the point of discontinuity. As it turns out, **no matter how large** n is, this error remains, and it has an amplitude of about 9% of the discontinuity. As n gets larger and larger, this wiggle becomes narrower and moves closer to the point of the discontinuity, but it **never goes away.** This overshoot phenomenon is known as the **Gibbs phenomenon**.

One of the important ramifications of this is in how we define functions to be equal. It is true that

$$\int_{T_0} (f(t) - \sum_{n=0}^{\infty} C_n \cos(n\omega_0 t + \theta_n))^2 dt = 0$$

that is, there is zero error between the function and the Fourier series, as defined by this squared integral criterion. But it does not mean that the functions are point-for-point equal. In this case, the error region simply becomes so small that the integral is zero. But this does not mean that

$$f(t) = \sum_{n=0}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

at every point. The mathematicians who like to leave no such stones unturned have made this an object of tremendous study, and consider such qualifications as equal "almost everywhere" (a.e.) or equal "with probability one". These are both in distinction to equal "everywhere" or equal "always." We have to be careful what we mean when we say two things are equal! (Ayn Rand would probably have trouble with this: perhaps it is not the case that $A = A$ is always true!)

Using Fourier Series for Signal Analysis

Suppose we have a system with transfer function $H(s)$ and a periodic input signal $f(t)$. What is the output signal? One way to do this, of course, is to convolve the input signal with the impulse response. But we all know how much we love convolution, and there is not a lot of insight to be gained from such a brute force computation. Another approach is to represent $f(t)$ in terms of sinusoids, then use the properties of L.T.I. systems. Specifically, if

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n).$$

then the output will be the sum of the responses due to each input:

$$y(t) = C_0 H(0) + \sum_{n=1}^{\infty} C_n |H(jn\omega_0)| \cos(n\omega_0 t + \theta_n + \angle H(jn\omega_0))$$

Discuss what happens in terms of filtering. In the homework you will work an example of this.

Exponential Fourier Series

We have seen how sin and cosine functions can be represented in terms of complex exponentials. It turns out that we can use complex exponentials to represent Fourier series. In many respects, this makes for a simpler representation.

Let's go back to the compact Fourier series representation function, and express it in terms of complex exponentials:

$$C_n \cos(n\omega_0 t + \theta_n) = D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}$$

where

$$D_n = \frac{1}{2} e^{j\theta_n} C_n$$

$$D_{-n} = \frac{1}{2} e^{-j\theta_n} C_n$$

We can write the Fourier series:

$$\begin{aligned} f(t) &= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \\ &= D_0 + \sum_{n=1}^{\infty} D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t} \\ &= D_0 + \sum_{n \neq 0} D_n e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \end{aligned}$$

Now, how do we find the coefficients? First, note that if we define the inner product correctly, the exponentials are orthogonal. Up to now, inner products have been defined for real functions. We will extend this now to inner products over complex functions. If $f(t)$ and $g(t)$ are a periodic complex functions with period T_0, define the inner product as

$$\langle f(t), g(t)\rangle = \int_{T_0} f(t)\overline{g(t)}dt$$

where the over-line means complex conjugate. The rules for this inner product are the same as before, except that

$$\langle f, g\rangle = \overline{\langle g, f\rangle}$$

With this inner product, note that

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t}\rangle = \int_{T_0} e^{jn\omega_0 t}e^{-jm\omega_0 t}dt = \frac{1}{j(n-m)\omega_0}[e^{j(n-m)2\pi} - 1]$$

which is 0 if $n - m \neq 0$ and T_0 is $n = m$. So under this inner product, we have a whole set of orthogonal functions. The geometry of orthogonal functions we talked about before applies, including the orthogonality theorem. We can therefore write

$$D_n = \text{proj}(f(t), e^{jn\omega_0 t}) = \frac{\langle f, e^{jn\omega_0 t}\rangle}{\langle e^{jn\omega_0 t}, e^{jn\omega_0 t}\rangle} = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}dt$$

(Derive this another way also.)

Note that this is true for all values of n (there are no special cases when $n = 0$) and there is only one formula (not two, as for sins and cosines). This is my preferred form! In fact, due to its similarity with the Fourier transform to be discussed soon, it is the most common form of the Fourier series. It is, of course, possible to convert from one form to another. For example,

$$D_n = \frac{1}{2}(a_n - jb_n)$$

Suppose (as is most often the case) that $f(t)$ is a real function. Then

$$\overline{D_n} = \overline{\frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}dt} = \frac{1}{T_0}\int_{T_0} f(t)e^{jn\omega_0 t}dt = D_{-n}.$$

Example 7 Find the exponential F.S. of the square wave function with period 2π of p. 428.

$$D_n = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jnt}dt = \frac{1}{2}\frac{\sin(n\pi/2)}{n\pi/2}$$

This is a good time to introduce a function that is near and dear to the heart of engineers:

$$\text{sinc}(x) = \frac{\sin x}{x}$$

So we can write

$$D_n = \frac{1}{2}\,\text{sinc}(n\pi/2)$$

Note: sinc(0) = 1. (How do we know?) □

Important observation: To compute the F.S. coefficients, multiply the function by an exponential with *negative* exponent. There are many transforms that electrical engineers use — Laplace transforms, Z-transforms, Fourier series, Fourier transforms, etc. In *all* of these, the exponent is negative. Don't forget it!

Exponential Fourier Spectra

As for the trigonometric F.S. we can make a plot of the F.S. by plotting the magnitude and phase of the complex numbers. For the example above,

$$D_0 = 1/2 \quad |D_0| = 1/2 \quad \angle D_0 = 0$$

$$D_1 = 1/\pi \quad |D_1| = 1/\pi \quad \angle D_1 = 0$$

$$D_{-1} = 1/\pi \quad |D_{-1}| = 1/\pi \quad \angle D_{-1} = 0$$

$$D_2 = 0 \quad D_{-2} = 0$$

When plotting the spectrum, both positive and negative values of n need, in general, to be plotted.

Bandwidth of a signal

The bandwidth of a signal is the amount of "frequency" required to sustain the signal unmodified. Actually, there are about a bajillion different definitions for bandwidth, depending on the application of the problem.

Example 8 Spectra on p. 445. The bandwidth is the highest nonzero frequency − the lowest nonzero frequency = 9 − 0 = 9.

□

Example 9 The bandwidth of the square wave function is infinite! What is often done from a practical power of view is to go out "far enough" – till the terms not included are small enough to worry about. Just what is "far enough" depends on the particular application. □

Example 10 In this example, we will find the F.S. of an important function, the periodic set of pulses. This is used (as we will see in chapter 8) as a representation of the sampling process.

$$\delta_{T_0}(t) = \sum_n \delta(t - nT_0)$$

Plot and explain. This is clearly periodic, and hence has a F.S. (in some sense). We can write

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where we can compute the coefficients from

$$D_n = \frac{1}{T_0}\int_{T_0} \delta_{T_0}(t) e^{-jn\omega_0 t} dt$$

Taking the integral from $-T_0/2$ to $T_0/2$ we get

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0}$$

This gives the important formula

$$\delta_{T_0}(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

The spectrum does not decay! $|D_n| = \frac{1}{T_0}$ and $\angle D_n = 0$. □

Energy of signals and Parseval's relationships

It is possible and often theoretically useful to examine the energy of signals in both the time domain and the frequency (Fourier series) domain. We will develop an important relationship. Suppose $f(t)$ is a periodic function with F.S. representation

$$f(t) = \sum_n D_n e^{jn\omega_0 t}$$

and $g(t)$ is a periodic function with the same period and a F.S. representation

$$g(t) = \sum_n E_n e^{jn\omega_0 t}$$

Now consider an average energy kind of term

$$\frac{1}{T_0}\int_{T_0} f(t) g^*(t) dt$$

Substituting in for each of the F.S. gives (taking advantage of the orthogonality of the exponential function)

$$\frac{1}{T_0}\int_{T_0} f(t) g^*(t) dt = \sum_n D_n E_n^*$$

We can write this in a convenient inner product notation. We can define the inner product between two series $\{D_n\}$ and $\{E_n\}$ as

$$\langle D_n, E_n\rangle = \sum_n D_n E_n^*$$

Then we can write (using our complex inner product for functions)

$$\frac{1}{T_0}\langle f(t), g(t)\rangle = \langle D_n, E_n\rangle$$

A relationship such as this is known as a *Parseval's* relationship, named after some guy.

As a special case, take $g(t) = f(t)$. Then $\frac{1}{T_0}\langle f(t), f(t)\rangle$ is the average energy of $f(t)$. By the Parseval's relationship,

$$\frac{1}{T_0}\langle f(t), f(t)\rangle = \langle D_n, D_n\rangle$$

Example 11 Find the sum of the series

$$\sum_n \left(\frac{\pi}{2}\operatorname{sinc}(n\pi/2)\right)^2$$

By recognizing the terms from the F.S. for a square wave with period 2π, we can use Parseval's relationship:

$$\sum_n \left(\frac{\pi}{2}\operatorname{sinc}(n\pi/2)\right)^2 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)^2 dt = \frac{1}{2}$$

This would have been hard to do any other way! □

A return to the geometric viewpoint

We have seen that functions can be represented as series of orthogonal functions, and have seen examples of orthogonal functions, the trigonometric and the complex exponentials. Historically, these were first examined by Jean Baptiste Fourier, who used these to solve the partial differential equations related to heat flow. At first his methods were considered unconventional by mathematicians. Now the generalization of Fourier's methods form one of the largest and most fruitful areas of mathematics.

Are there any other useful orthogonal functions? Given a set of functions that are not orthogonal, is it possible to make it orthogonal somehow? Both answers are yes.

We begin looking at a set of orthogonal polynomials, defined over the interval $[-1, 1]$. It is easy to check that the set of polynomials $\{1, t, t^2, t^3, \ldots\}$ is not orthogonal. For example,

$$\langle t, t^3\rangle = \int_{-1}^{1}(t)(t^3)dt = \frac{2}{5}$$

Consider, however, the polynomials

$$P_0(t) = 1$$

$$P_1(t) = t$$

$$P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$$

$$P_3(t) = \frac{5}{2}t^3 - \frac{3}{2}t$$

In general,

$$P_n = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2 - 1)^n.$$

These polynomials are known as the *Legendre* polynomials. It can be shown that

$$\int_{-1}^{1} P_m(t)P_n(t)dt = \begin{cases} 0 & m \neq n \\ \frac{2}{2m+1} & m = n \end{cases}$$

For functions that are defined over any finite interval, Legendre polynomials can be used as a functional representation.

Another way that orthogonal functions arise is by means of another inner product. While we have not mentioned it in the past, it is possible to introduce a positive weighting factor into the inner product. Every one of these produces a new inner product, each with properties that may be useful for particular applications. If $w(t)$ is a non-negative function, then we can define an inner product

$$\langle f, g \rangle_w = \int_I w(t)f(t)g(t)dt$$

where the subscript $_w$ indicates the weighting and I is some interval of integration. For example, for $w(t) = \frac{1}{\sqrt{1-t^2}}$ and $I = [-1, 1]$ we might define an inner product as

$$\langle f(t), g(t) \rangle_w = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)g(t)dt$$

A set of polynomials that is orthogonal with respect to this norm is the set of *Chebyshev polynomials*:

$$T_0(t) = 1$$

$$T_n(t) = \cos n \cos^{-1} t$$

For example,

$$T_1(t) = t$$

$$T_2(t) = \cos 2\cos^{-1} t = 2\cos^2(\cos^{-1} t) - 1 = 2t^2 - 1$$

$$T_3(t) = \cos 3\cos^{-1} t = 4\cos^3(\cos^{-1} t) - 3\cos(\cos^{-1} t) = 4t^3 - 3t$$

Notice that these have the "equal ripple" property. This makes them useful in function approximation (and in fact, Chebyshev functions are used at the heart of the Remez algorithm).

Another set of interesting orthogonal functions that have been the topic of an incredible amount of research lately are *wavelet functions*. Unlike the familiar and common trigonometric functions, there are actually several families of wavelets (this is part of what makes it confusing). One type of wavelets is described by two sets of functions: $\phi(t)$ (known as a scaling function), and $\psi(t)$ (known as a wavelet function). In dealing with these functions, instead of looking at different frequencies, we *scale* and *shift* these functions. We define

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$$

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$$

Make plots and explain. Then (somewhat miraculously), the following orthogonality properties exist:

$$\langle \phi_{j,k}(t), \phi_{j,m}(t) \rangle = \delta_{k,m}$$

$$\langle \psi_{j,k}(t), \psi_{l,m}(t) \rangle = \delta_{j,l}\delta_{k,m}$$

$$\langle \phi_{j,k}(t), \psi_{l,m}(t) \rangle = \delta_{j,l}\delta_{k,m}$$

These are pretty remarkable! This gives us a whole bunch of functions that we can use to signal representations:

$$f(t) = \sum_k a_{J,k}\phi_{J,k}(t) + \sum_{j,k} b_{j,k}\psi_{j,k}(t)$$

This is useful for a variety of things which will become more clear after we talk about Fourier transforms (next chapter). But notice that we can represent *any* function, not just periodic functions, localizing both frequency information and time information. There are also very efficient algorithms that are faster than the Fast Fourier Transform (FFT) to compute a discrete wavelet transform.