ECE 3640
Lecture 11 – State-Space Analysis

Objective: To learn about state-space analysis for continuous and discrete-time systems

Perspective

Transfer functions provide only an input/output perspective of what is going on in a system. There may be things going on physically that do not appear in a transfer function, due to cancellations, etc. On the other hand, state-space analysis provides a more complete representation. Furthermore, it can be generalized to time-varying systems, multi- input or output systems, and in some applications leads to very explicit design formulations. There is also much that can be done with nonlinear systems in state variable form.

We have seen that we can describe an LTIC system using a single differential equation. In state-space analysis, we deal with *systems* of equations, but make it so that all equations are *first order*. Sometimes this requires introducing some extra variables. The variables appearing in these equations (with respect to which we differentiate) are called the *state* variables. The idea behind the name is this: for a first order differential equation, if we know where we are initially (the initial condition), then this provides all of the information we need to determine where to go.

In circuits, it is common to choose the voltage across the capacitors and the current through the inductors as state variables. This provides our first example.

Example 1 Circuit. Two state variables.

KCL:

$$0.2\dot{x}_1 = i_1 - i_2 - x_2$$

Ohm's: $i_1 = 2(f - x_1)$. $i_2 = 3x_1$. We obtain

$$0.2\dot{x}_1 = 2(f - x_1) - 3x_1 - x_2$$

or

$$\dot{x}_1 = -25x_1 - 5x_2 + 10f$$

Notice that everything is expressed in terms of the state variables x_1 and x_2 and the input f.

Next equation: KVL:

$$(1)\dot{x}_2 + 2i_4 - x_1 = 0.$$

since $i_4 = x_2$ we obtain

$$\dot{x}_2 = x_1 - 2x_2.$$

We obtain

$$\begin{aligned}\dot{x}_1 &= -25x_1 - 5x_2 + 10f \\ \dot{x}_2 &= x_1 - 2x_2.\end{aligned}$$

Note: every possible output of the circuit — every voltage and current — can be expressed in terms of the state variables. Try a few. Express in matrix form. □

Note: we have

1. First order differential equations.
2. Each equation is expressed in terms of the state variables and the input.

For *linear* equations, we can put the equations in matrix form. Let

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Let $\dot{\mathbf{x}}$ denote taking the derivatives individually:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$$

In the example above, let

$$A = \begin{bmatrix} -25 & -5 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

Then we can write

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bf(t).$$

Note: For nonlinear systems, we can still put them in state variable form, even when we cannot use a matrix for the representation.

Example 2

$$\begin{aligned}\dot{x}_1 &= x_2^2 + x_3 \cos(x_1 + x_2) + f^2 \\ \dot{x}_2 &= (x_1 + x_2)^2 \\ \dot{x}_3 &= x_3 \tan(x_2/x_1)\end{aligned}$$

□

Example 3 (Important) Consider the 3rd order equation

$$(D^3 + a_2D^2 + a_1D + a_0)y(t) = f(t).$$

We will introduce some new variables. Let

$$\begin{aligned}x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y}\end{aligned}$$

Then

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= f(t) - a_2\dot{x}_3 - a_1\dot{x}_2 - a_0\dot{x}_1 + f\end{aligned}$$

We also have an output equation

$$y = x_1.$$

We can stack this as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}f$$

$$y = \mathbf{c}^T\mathbf{x}.$$

□

In general, a set of state-variable equations can be written

$$\dot{x}_i = g_i(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_j), \qquad i = 1, 2, \ldots, n$$

$$y_i = h_i(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_j), \qquad i = 1, 2, \ldots, k$$

Note that this could be

- Nonlinear
- Multiple inputs
- Multiple outputs

(But may not be!) This represents a considerable degree of flexibility.

A general j-input, k-output *linear* system with n state variables can be written as

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = A\dot{\mathbf{x}} + B\mathbf{f}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{f}$$

where A is $n \times n$, B is $n \times j$, C is $k \times n$ and D is $k \times j$. (Write out the matrices.)

Have a student work the circuit on the board. .

Transfer functions and state equations

Given a transfer function, we may want to write a state variable equation for it. This is very straightforward by writing a system realization for the transfer function. From the system realization, we **let the state variables be the outputs of the integrators**.

Example 4

$$H(s) = \frac{2s + 10}{s^3 + 8s^2 + 19s + 12}$$

Canonical Realization:

State equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -12x_1 - 19x_2 - 8x_3 + f \end{aligned}$$

Output equation: $y = 2x_2 + 10x_1$.

Matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f$$

$$y = \begin{bmatrix} 10 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Comment on the form of the matrix: **companion matrix**. Describe form in general. What is the characteristic equation of the companion matrix? What are the eigenvalues?

Observer Form Realization: Write equations, then in matrix form. Companion

matrix. What is the characteristic equation? Eigenvalues?

Series form Realization: Equations:

$$H(s) = \frac{2}{s+1}\frac{s+5}{s+3}\frac{1}{s+4}$$

$$\begin{aligned} \dot{w}_1 &= -w_1 + f \\ \dot{w}_2 &= 2w_1 - 3w_2 \\ \dot{w}_3 &= 5w_2 + \dot{w}_2 - 4w_3 \end{aligned}$$

Eliminate $\dot{w}_2$ using the 2nd eqn. Obtain

$$\dot{\mathbf{w}} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 2 & 2 & -4 \end{bmatrix} \mathbf{w} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} f$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{w}.$$

Characteristic eqn? Eigenvalues?

Parallel Realization:

Equations:

$$H(s) = \frac{4/3}{s+1} - \frac{2}{s+3} + \frac{2/3}{s+4}$$

$$\begin{aligned} \dot{z}_1 &= -z_1 + f \\ \dot{z}_2 &= -3z_2 + f \\ \dot{z}_3 &= -4z_3 + f \end{aligned}$$

$$y = \frac{4}{3}z_1 - 2z_2 + \frac{2}{3}z_3$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} f$$

$$y = \begin{bmatrix} \frac{4}{3} & -2 & \frac{2}{3} \end{bmatrix} \mathbf{z}$$

Characteristic polynomial? Eigenvalues? □

Note: There are other ways of writing down the state equations from the transfer function. In fact, there are an infinite number of ways!

Laplace transform of state equations

When we talk about the Laplace transform of a vector, we will mean to apply the transform element by element. Thus, if

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

then

$$\mathcal{L}[\mathbf{x}(t)] = \begin{bmatrix} \mathcal{L}[x_1(t)] \\ \mathcal{L}[x_2(t)] \end{bmatrix} = \mathbf{X}(s).$$

We find then that

$$\mathcal{L}[\dot{\mathbf{x}}(t)] = s\mathbf{X}(s) - \mathbf{x}(0).$$

From the state equation $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{f}$ we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + B\mathbf{F}(s)$$

and from the output equation,

$$\mathbf{Y}(s) = C\mathbf{X}(s) + D\mathbf{F}(s).$$

Let us solve for $\mathbf{X}(s)$ from the first:

$$(sI - A)\mathbf{X}(s) = \mathbf{x}(0) + B\mathbf{F}(s)$$

(Why the identity?) Watch the order!

$$\mathbf{X}(s) = (sI - A)^{-1}[\mathbf{x}(0) + B\mathbf{F}(s)].$$

Let $\Phi(s) = (sI - A)^{-1}$. We have

$$\mathbf{X}(s) = \Phi(s)[\mathbf{x}(0) + \Phi(s)B\mathbf{F}(s).]$$

Inverse transform:

$$\mathbf{x}(t) = \mathcal{L}^{-1}[\Phi(s)\mathbf{x}(0)] + \mathcal{L}^{-1}[\Phi(s)B\mathbf{F}(s)].$$

Identify zero-input components and zero-state components.

Output:

$$Y(s) = C\Phi(s)\mathbf{x}(0) + [C\Phi(s)B + D]F(s)$$

Transfer function:

$$H(s) = C\Phi(s)B + D$$

Example 5

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

(Two inputs!)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

(Three outputs!) Identify A,B,C,D.

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -12 & s+3 \end{bmatrix}^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Transfer function:

$$H(s) = \begin{bmatrix} \frac{s+4}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2(s-2)}{(s+1)(s+2)} & \frac{2s}{(s+1)(s+2)} \end{bmatrix} + D = \begin{bmatrix} \frac{s+4}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{2(s+3)}{s+2} & \frac{1}{s+2} \\ \frac{2(s-2)}{(s+1)(s+2)} & \frac{2(s^2+4s+2)}{(s+1)(s+2)} \end{bmatrix}$$

□

Poles and Eigenvalues

Recall:

$$X^{-1} = \frac{\text{adj}(X)}{\det(X)}$$

Without worrying about what the adj is, note that the denominator always has the determinant. Thus

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}.$$

So the denominator has poles where the eigenvalues of A are!

Time domain solution

We begin by defining a new function. For a square matrix A (as in the state transition matrix) we define

$$e^A = (I + A + \frac{a^2}{2} + \frac{A^3}{3!} + \cdots)$$

(Taylor series). This is directly analogous to e^a for scalars, except that all arithmetic is done using matrices. This is computed using the `exmp` function in Matlab, not `exp`. Note (show this)

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A.$$

The solution to the DE

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{f}$$

is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}Bf(\tau)\,d\tau$$

Show that it works by substitution.

Computing the matrix exponential: One way:

$$e^{At} = \mathcal{L}^{-1}\Phi(s) = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

Example 6 $A = \begin{bmatrix} -12 & 2/3 \\ -36 & -1 \end{bmatrix}$.

$$(sI - A)^{-1} = \begin{bmatrix} s+12 & -2/3 \\ 36 & s+1 \end{bmatrix}^{-1} = \frac{1}{(s+12)(s+1)+24}\begin{bmatrix} s+1 & 2/3 \\ -36 & s+12 \end{bmatrix} = \frac{1}{(s+4)(s+9)}\begin{bmatrix} s+12 & -2/3 \\ 36 & s+1 \end{bmatrix}.$$

Taking inverse Laplace transforms element by element we obtain

$$e^{At} = \begin{bmatrix} -0.6e^{-9t} + 1.6e^{-4t} & 0.133e^{-9t} - 0.133e^{-4t} \\ -7.2e^{-9t} + 7.2e^{-4t} & 1.6e^{-9t} - 0.6e^{-4t} \end{bmatrix}$$

□

Linear transformations

For the state equations

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{f}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{f}$$

let us create a new variable $\mathbf{w} = P\mathbf{x}$ for an invertible matrix P. Then $\mathbf{x} = P^{-1}\mathbf{w}$, and $\dot{\mathbf{x}} = P^{-1}\dot{\mathbf{w}}$. Substituting we find

$$P^{-1}\dot{\mathbf{w}} = AP^{-1}\mathbf{w} + B\mathbf{f}$$

or

$$\dot{\mathbf{w}} = PAP^{-1}\mathbf{w} + PB\mathbf{f} = \hat{A}\mathbf{w} + \hat{B}\mathbf{f}$$

where

$$\hat{A} = PAP^{-1} \qquad \hat{B} = PB.$$

Similarly,

$$\mathbf{y} = \hat{C}\mathbf{w} + \hat{D}\mathbf{f}$$

where

$$\hat{C} = CP^{-1} \qquad \hat{D} = D.$$

Instead of (A, B, C, D) we have $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$.

Do these represent the same system?

$$H(s) = C(sI - A)^{-1}B + D$$

$$\hat{H}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}.$$

(Work through details.)

Other observations: eigenvalues? Eigenvectors?

0.1 A special transformation: diagonalizing A

Given $\hat{A} = PAP^{-1}$, suppose that we want to find a transformation matrix P such that $\hat{A}$ is diagonal. (This is a convenient form, since it "decouples" all the modes.) How can we find such a P?

Let $\mathbf{e}_i$ be eigenvectors of A, and λ_i be the eigenvalues of A, assumed (for our purposes) to be unique. Form

$$Q = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$$

and

$$AQ = Q\Lambda$$

where

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Then

$$\Lambda = Q^{-1}AQ$$

Identify: $\Lambda = \hat{A}$, $P = Q^{-1}$.

1 Controllability and observability

Example 7 Cascade representation

$$H_1(s) = \frac{1}{s-1}\frac{s-1}{s+1}$$

State variable form:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad d = 0$$

$$\det(sI - A) = (s+1)(s-1)$$

Eigenvalues: ± 1. Diagonalize:

```
A = [1 0; 1 -1];
b = [1;0];
c = [1 -2];
[u,v] = eig(A);  % u has eigenvectors, v = eigenvalues
Q = u;
% Check:
inv(Q)*A*Q       % should be diagonal!
P = inv(Q)
Ahat = P*A*inv(P)
bhat = P*b
chat = c*inv(P)
```

We find

$$\hat{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \hat{\mathbf{b}} = \begin{bmatrix} -.5 \\ 1.118 \end{bmatrix} \qquad \hat{\mathbf{c}}^T = [-2, 0]$$

Write state equations. Second state variable: not **observable**.

Now the second system:

$$A = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

As before, diagonalize:

```
A = [-1 0; -2 1];
b = [1;1];
c = [0 1];
[u,v] = eig(A);  % u has eigenvectors, v = eigenvalues
```

```
Q = u;
% Check:
inv(Q)*A*Q       % should be diagonal!
P = inv(Q)
Ahat = P*A*inv(P)
bhat = P*b
chat = c*inv(P)
```

We find

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \hat{b} = \begin{bmatrix} 0 \\ 1.4142 \end{bmatrix} \qquad \hat{\mathbf{c}}^T = [1, 0.7071].$$

Write state equations. Second state variable not **controllable**. □

Note that in both cases, the end-to-end transfer function hides some information — there is cancellation there. The transfer function provides a potentially inadequate representation of what is going on.

In the general case, let us write

$$\dot{\mathbf{z}} = \Lambda \mathbf{z} + \hat{B}\mathbf{f}$$

$$\mathbf{y} = \hat{C}\mathbf{z} + \hat{D}\mathbf{f}$$

where Λ is a **diagonal** matrix — all the modes uncoupled. If there is a row of zeros in B, then $\mathbf{f}$ has no influence on the corresponding state variable. That variable is said to be **uncontrollable**. If there is a column of zeros in $\hat{C}$, then the corresponding state variable is said to be **unobservable**. For many purposes, systems should be both controllable and observable.

Discrete-time

Most of what can be said for continuous time can also be said for discrete time:

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + B\mathbf{f}[k]$$

$$\mathbf{y}[k] = C\mathbf{x}[k] + D\mathbf{f}[k].$$

Solution:

$$\mathbf{x}[k] = A^k \mathbf{x}[0] + \sum_{j=0}^{k-1} A^{k-1-j} B\mathbf{f}[j].$$

(Show how this works by recursion), starting from

$$\mathbf{x}[1] = A\mathbf{x}[0] + B\mathbf{f}[0].$$

Z-transform:

$$zX(z) - z\mathbf{x}[0] = AX(z) + BF(z).$$

$$X(z) = (zI - A)^{-1}[z\mathbf{x}[0] + BF(z)]$$

$$H(z) = C(zI - A)^{-1}B + D$$