ECE 3640
Lecture 1 – Discrete-time systems

Objective: To learn about discrete-time systems. To learn about numeric solution of difference equations. To learn about analytical solution of difference equations, including the zero-input response and the zero-state response. To investigates stability issues for discrete-time systems.

Reading: pp. 540–616

Now we are ready to make a change of direction. Up till now we have focused on continuous time systems. Now we will look at discrete-time systems. This (I hope) will reinforce some of the stuff we have seen.

Where do these things come from? (C/D, discrete-time system, D/C). There is a sample interval T. Many systems have an intrinsically defined period: days, weeks, months, etc. It is common to write $y(kT) = y[k]$. (I may get sloppy on the parentheses and the brackets.)

Example 1 A sampled signal. suppose $f(t) = A\cos(\omega_0 t + \theta_0)$, and we sample every T seconds:

$$f[k] = f(kT) = A\cos(\omega_0 Tk + \theta_0)$$

Let $\Omega_0 = \omega_0 T$.

$$f[k] = A\cos(\Omega_0 k + \theta_0).$$

We get a new frequency, Ω_0, whose units are in radians per sample. The amplitude does not change; the phase does not change.

Now consider a specific example. Suppose $\omega_0 = 2\pi 1000$ (A 1000-Hz signal). Let $T = \frac{1}{4000}$. We get

$$f[k] = A\cos(2\pi 1000(1/4000)k) = A\cos(\pi/2k).$$

so $\Omega_0 = \pi/2$. (Plot this.)

Now let $T = \frac{1}{1000}$. $f[k] = 1$. What happened?

Now let $T = \frac{1}{1000} - \frac{1}{4000}$. What happens? □

Example 2 An example of a discrete-time Bank deposit $= f[k]$. Bank balance $= y[k]$. Bank pays some interest every period r. Then the money goes as

$$y[k] = y[k-1] + ry[k-1] + f[k]$$

(The money at the end of the k period is the money that was there, plus the money due to interest, plus any new deposits). Draw the block diagram (realization). □

Example 3 If $y(t) = \frac{df}{dt}$, then

$$y(kT) = \frac{df}{dt}|_{t=kT}$$

$$y[k] \approx \frac{1}{T}(f[k] - f[k-1])$$

□

Instead of using RLC, the elements in discrete-time systems are **delays, adders, and multipliers.**

Comment on advantages: precision, stability, flexibility, variety, size, storage reliability, sophistication, sharing, cost.

Some useful signal models for discrete-time

Discrete-time unit impulse $\delta[k]$:

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Discrete-time unit step $u[k]$:

$$u[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Discrete-time exponential γ^k.

γ^k grows when $|\gamma| > 1$, and decays when $|\gamma| < 1$. Plot on unit circle.

What about when $|\gamma| = 1$? We can write this as $e^{j\Omega k}$ for some Ω. This is a rotating phasor.

Discrete-time sinusoid $\cos(\Omega k + \theta)$.

Note: Not all discrete-time sinusoids are periodic (unlike in the continuous-time case):

$$f[k] = f[k + N_0]$$

This is only possible if ΩN_0 is an integral multiple of 2π:

$$\cos(\Omega k) = \cos(\Omega(k + N_0))$$

iff

$$\Omega N_0 = 2\pi m$$

or

$$\frac{\Omega}{2\pi} = \frac{m}{N_0}$$

Must therefore have $\Omega/2\pi$ be a **rational** number.

Note: There is non-uniqueness of discrete-time sinusoids. For example:

$$\cos(9.6\pi k + \theta) = \cos(8\pi k + 1.6\pi k + \theta) = \cos(1.6\pi k + \theta) = \cos(-1.4\pi k + \theta) = \cos(0.4\pi k - \theta)$$

All these frequencies "look" the same!

Exponentially damped sinusoid $\gamma^k \cos(\Omega k + \theta)$

Some useful signal operations

Essentially the same as for continuous time:

Time shifting: $f[k + m]$ (shift left by m), etc.

Time reversal $f[-k]$

Time compression: Decimation

$$g[k] = f[2k]$$

(take every other point)

Time expansion

$$g[k] = f[k/2]$$

when $k/2$ is an integer.

Difference equations

The equations that arise in discrete-time systems are not differential equations, but *difference* equations. We could write

$$y[k+n]+a_{n-1}y[k+n-1]+\cdots a_1 y[k+1]+a_0 y[k] = b_m f[k+m]+b_{m-1}f[k+m-1]+\cdots+b_1 f[k+1]+b_0 f[k]$$

(advance operator form) It could also be written as (by $k \to k-n$)

$$y[k]+a_{n-1}y[k-1]+\cdots a_1 y[k-n+1]+a_0 y[k-n] = b_n f[k]+b_{n-1}f[k-1]+\cdots+b_1 f[k-n+1]+b_0 f[k-n]$$

(delay operator form).

Numerically it is straightforward to find solutions of the system equation given some initial conditions:

$$y[k] = -a_{n-1}y[k-1]-a_{n-2}y[k-2]-\cdots-a_0 y[k-n]+b_n f[k]+b_{n-1}f[k-1]+\cdots+b_0 f[k-n]$$

Just propagate forward.

Example 4 Fibonacci series: $y[1] = 1$, $y[2] = 1$, $y[n] = y[n-1] + y[n-2]$. □

Example 5 $y[k] - 0.5y[k-1] = f[k]$. Suppose we know $y[-1] = 16$ and $f[k] = k^2$.

$$y[k] = 0.5y[k-1] + f[k]$$

$$y[0] = 0.5(16) + 0 = 8$$
$$y[1] = 0.5(8) + 1 = 5$$
$$y[2] = 0.5(5) + 4 = 6.5$$

etc. □

We will use E to indicate the advance operator: $Ef[k] = f[k+1]$, etc.

Example 6 $y[k+2] + \frac{1}{4}y[k+1] + \frac{1}{16}y[k] = f[k+2]$. In operator notation:

$$(E^2 + \frac{1}{4}E + \frac{1}{16})y[k] = E^2 f[k]$$

(A lot like what we did for differential equations.) □

We can write the general nth order difference equation as

$$(E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0)y[k] = (b_n E^n + b_{n-1}E^{n-1} + \cdots + b_1 E + b_0)f[k]$$

or

$$Q[E]y[k] = P[E]f[k]$$

Now we do the same kinds of things we did before: the zero-input response, then the zero-state response.

Zero-input response

When there is no input, we can write

$$Q[E]y_0[k] = 0$$

or

$$(E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0)y_0[k] = 0$$

(What happened for continuous time?) Similar. Let's try a simply case to get started:

$$(E - \gamma)y_0[k] = 0$$

Try a solution $y_0[k] = c\gamma^k$. (Comment on the difference from earlier case.) Substitute in and show that it works.

$$Ec\gamma^k = c\gamma^{k+1}$$

This works in the general case: subs. and show that it works. Substituting gives

$$(\gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0)y_0[k] = 0$$

or

$$Q[\gamma]y_0[k] = 0$$

For an interesting (nontrivial) solution, we will look for roots of

$$Q[\gamma] = 0$$

Write as

$$(\gamma - \gamma_1)(\gamma - \gamma_2)\cdots(\gamma - \gamma_n) = 0$$

$Q[\gamma]$ is thus the characteristic polynomial, and we look at its roots. The roots are $\gamma_1, \gamma_2, \ldots, \gamma_n$.

As before, we take all possible solutions in a linear combination:

$$y_0[n] = c_1\gamma_1^k + c_2\gamma_2^k + \cdots + c_n\gamma_n^k$$

Repeated roots: If

$$Q[\gamma] = (\gamma - \gamma_1)^r(\gamma - \gamma_{r+1})(\gamma - \gamma_{r+2})\cdots(\gamma - \gamma_n)$$

then

$$y_0[k] = (c_1 + c_2k + \cdots c_rk^{r-1})\gamma_1^k + c_{r+1}\gamma_{r+1}^k + \cdots + c_n\gamma_n^k$$

So how do things behave as a function of root location. (0.8, -0.8, -.5, 1.2, $k(0.8)$, 1, etc.) Same old, same old.

Example 7 Solve

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

with $y[-1] = 0$ and $y[-2] = 25/4$ and $f[k] = 4^{-k}u[k]$.

$$(E^2 - 0.6E - 0.16)y[k] = 5E^2f[k]$$

Characteristic polynomial:

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8) = 0$$

$$y_0[k] = c_1(-0.2)^k + c_2(0.8)^k$$

At $k = -1$,

$$0 = -5c_1 + \frac{5}{4}c_2$$

At $k = -2$,

$$25/4 = 25c_1 + \frac{25}{16}c_2$$

So $c_1 = 1/5$ and $c_2 = 4/5$. (We will get the zero-state solution later.) Is this asymptotically stable? □

Complex roots: With roots $\gamma = |\gamma|e^{j\pm\beta}$, then

$$y[k] = c|\gamma|^k \cos(\beta k + \theta)$$

with c and θ arbitrary constants (same steps as before).

Example 8

$$(E^2 - 1.56E + .81)y[k] = (E + 3)f[k]$$

$y[-1] = 2$ and $y[-2] = 1$. Characteristic polynomial:

$$\gamma^2 - 1.56\gamma + .81 = 0$$

Factor:

$$(\gamma - .78 - j.45)(\gamma - .78 + j.45) = 0$$

Roots: $.78 \pm j.45$ Convert to polar:

$$\gamma_1 = .9e^{j\pi/6} \qquad \gamma_2 = .9e^{-j\pi/6}$$

Zero-input solution:

$$y_0[k] = c(.9)^k \cos(\pi/6k + \theta)$$

Need to find c and θ using the initial conditions. At $k = -1$

$$2 = \frac{c}{.9}\cos(-\pi/6 + \theta).$$

At $k = -2$:

$$1 = \frac{c}{.81}\cos(-\pi/3 + \theta)$$

Using $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ (p. 65),

$$2 = \frac{\sqrt{3}}{1.8}c\cos\theta + \frac{1}{1.8}c\sin\theta$$

$$1 = \frac{1}{1.62}c\cos\theta + \frac{\sqrt{3}}{1.62}\sin\theta$$

Solve for unknowns $c\cos\theta$ and $c\sin\theta$:

$$c\cos\theta = 2.308$$

$$c\sin\theta = -.397$$

$$\theta = -.17 \text{ rad} \qquad c = 2.34$$

□

Zero-state response

The steps are similar to what we did before: introduce delta function, then the impulse response, then the convolution sum.

Discrete time impulse function:

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

(Plot). We have the discrete-time impulse response $h[k]$:

$$Q[E]h[k] = P[E]\delta[k]$$

Then $h[k]$ is the solution when initial conditions are all zero:

$$h[-1] = h[-2] = \cdots = h[-n] = 0$$

We can find a numerical solution by substitution.

Example 9 $y[k] - 0.6y[k-1] - .16y[k-2] = 5f[k]$:

$$h[k] - .6h[k] - .16h[k-2] = 5\delta[k]$$

with $h[-1] = h[-2] = 0$.

$$h[0] = 5$$

$$h[1] = 3$$

etc. How long does this go on? Comment on IIR vs. FIR. A closed form solution may be obtained from

$$h[k] = A\delta[k] + y_0[k]u[k]$$

where $A = b_0/a_0$. □

Example 10 $y[k] - 0.6y[k-1] - 0.16y[k-2] = 5f[k]$

$$(E^2 - 0.6E - 0.16)y[k] = 5E^2 f[k]$$

$$y_0[k] = c_1(-0.2)^k + c_2(0.8)^k$$

$$h[k] = 0 + [c_1(-0.2)^k + c_2(0.8)^k]u[k]$$

Need to find constants. At $k = 0$,

$$h[0] - 0.6h[-1] - 0.16h[-2] = 5$$

$$h[1] - 0.6h[0] - 0.16h[-1] = 0$$

$h[0] = 5, h[1] = 3 \rightarrow c_1 = 1, c_2 = 4$. □

Zero-state response: Observe that we can write

$$f[k] = f[0]\delta[k] + f[1]\delta[k-1] + \cdots = \sum_m f[m]\delta[k-m]$$

Now we add up the response of the system to each of these outputs:

$$f[0]\delta[k] \rightarrow f[0]h[k]$$

$$f[1]\delta[k-1] \rightarrow f[1]h[k-1]$$

Adding these up,

$$y[k] = \sum_m f[m]h[k-m]$$

Same sorts of properties as we had before:

Commutative $f_1[k] * f_2[k] = f_2[k] * f_1[k]$

Distributive $f_1[k] * (f_2[k] + f_3[k]) = f_1[k] * f_2[k] + f_1[k] * f_3[k]$

Associative $f_1[k] * (f_2[k] * f_3[k]) = (f_1[k] * f_2[k]) * f_3[k]$

Shifting $f_1[k-m] * f_2[k-n] = c[k-m-n]$

Convolution. with impulse $f[k] * \delta[k] = f[k]$

Width: If f_1 has length m points and f_2 has length n points, then $f_1 * f_2$ has length $m+n-1$ points. (Note length given in **points**.)

For causal systems with causal inputs

$$y[k] = \sum_{m=0}^{k} f[m]h[k-m]$$

Example 11 Find $c[k] = f[k] * g[k]$ when $f[k] = (0.8)^k u[k]$ and $g[k] = (0.3)^k u[k]$:

$$c[k] = \sum_{m=0}^{k} (0.8)^m (0.3)^{k-m}$$

$$c[k] = (0.3)^k \sum_{m=0}^{m} \left(\frac{0.8}{0.3}\right)^m$$

Important fact: emblazon this in your head. (Also be aware that symbolic packages know this.) The sum of a geometric series is

$$\boxed{\sum_{m=0}^{k} r^m = \frac{r^{k+1}-1}{r-1} \qquad r \neq 1}$$

We get

$$c[k] = (0.3)^k \frac{(0.8/0.3)^{k+1}-1}{(0.8/0.3)^k - 1} = 2[(.8)^{k+1} - (.3)^{k+1}]u[k]$$

□

Example 12 The problem we have seen before: $y[k+2] - .6y[k+1] - .16y[k] = 5f[k+2]$. with $f[k] = 4^{-k}u[k]$.

$$h[k] = [(-.2)^k + 4(.8)^k]u[k]$$

$$y[k] = h[k] * f[k].$$

Use tables (and a lot of simplification)

$$\begin{aligned} y[k] &= \left[\frac{(.25)^{k+1} - (-.2)^{k+1}}{.25 - (-.2)} + 4\frac{(.25)^{k+1} - (.8)^{k+1}}{.25 - .8}\right] u[k] \\ &= [-5.05(.25)^{k+1} - 2.22(-.2)^{k+1} + 7.27(.8)^{k+1}]u[k] \\ &= [-1.26(.25)^k + .444(-.2)^k + 5.81(.8)^k]u[k] \end{aligned}$$

□

Total response = zero-input component + zero-state component.

Example 13 Take the same system as before, with $y[-1] = 0, y[-2] = 25/4$. Then

$$y[k] = (.2(-.2)^k + .8(.8)^k) + (-1.26(.25)^k + .444(-.2)^k + 5.81(.8)^k)$$

□

Natural and forced response

The total solution may be divided into a natural and forced response, just like we did before. The natural response is those components of the total response which have the natural modes of the system. The forced response is everything else (which must necessarily come from the forcing function).

For the last example,

$$y[k] = .644(-.2)^k + 6.61(.8)^k - 1.26(.25)^k$$

System stability

Plots as a function of pole location. (Asymptotically) stable, unstable, marginally stable.

BIBO stability. System response to bounded inputs:

$$y[k] = h[k] * f[k]$$

$$\begin{aligned} |y[k]| &= \left| \sum h[m] f[k-m] \right| \\ &\leq \sum_m |h[m]||f[m-k]| \end{aligned}$$

For bounded input, $|f[k-m]| < K_1 < \infty$

$$|y[k]| \leq K_1 \sum_m |h[m]|$$

Bounded if $\sum_m |h[m]| < \infty$, or all roots inside unit circle.