Review of Finite Dimensional Vector Spaces

1. **Definition:** A *vector space* is a set V with $+$ and $\cdot$.

$$\begin{array}{l}\mathbf{x}, \mathbf{y} \in V \\ a, b \in \mathbb{R}\end{array} \quad \Longrightarrow \quad a \cdot \mathbf{x} + b \cdot \mathbf{y} = \mathbf{z} \in V$$

2. **Definition:** *Linear Combination*

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N \qquad \text{or} \qquad \mathbf{y} = \sum_{i=1}^{N} a_i\mathbf{x}_i$$

3. **Definition:** $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ *spans* V, if

$$\mathbf{y} = \sum_{i=1}^{N} a_i\mathbf{x}_i \quad \forall \ \mathbf{y} \in V$$

4. **Definition:** $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ are *linearly independent* if

$$0 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N \quad \Longrightarrow \quad a_1 = a_2 = \cdots = a_N = 0$$

   Otherwise they are *linearly dependent*.

5. **Definition:** $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ is a *basis* for V if (1) they span V and (2) are linearly independent.

6. **Theorem:** Let $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ be a basis for V, then

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N$$

   is unique.

7. **Definition:** $(a_1, a_2, \cdots, a_N)$ are the coordinates of $\mathbf{y}$ with respect to the given basis for V.

8. **Definition:** The *dimension* of V is the number of elements in a basis for V.

9. **Definition:** Let us construct the unique $N \times 1$ *coordinate vector* corresponding to the element $\mathbf{y}$ in V.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

   The coordinate vector "lives" in N-dimensional Euclidean space regardless of the nature of the vector space V.

Operations on Vectors in Euclidean Space

Let V be an N-dimensional Euclidean space and let $\mathbf{a}, \mathbf{b}$ be elements of V.

1. **Definition:** *Inner Product*

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{N} a_i b_i = \langle \mathbf{b}, \mathbf{a} \rangle \qquad \text{Measures how alike or parallel } \mathbf{a} \text{ and } \mathbf{b} \text{ are.}$$

2. **Definition:** *Angle* between two vectors $\mathbf{a}, \mathbf{b}$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\theta$$

3. **Definition:** $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal* if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$.

4. **Definition:** *Norm* of a vector $\mathbf{a}$

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\sum_{i=1}^{N} a_i^2} \qquad \text{Measures the length of } \mathbf{a}.$$

$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \qquad \langle \mathbf{a}, \mathbf{b} \rangle \leq |\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|$$

5. **Definition:** *Unit Vector*

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \qquad \|\mathbf{u}\| = 1$$

6. **Definition:** *Distance* between two points $\mathbf{a}, \mathbf{b}$

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| = \sqrt{\sum_{i=1}^{N} (a_i - b_i)^2}$$

7. **Definition:** $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N$ are *orthonormal* if

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} = \delta_{i-j} \qquad \text{(Kronecker delta)}$$

Geometric Representation of Signals

Let V be an N-dimensional vector space of finite energy signals of duration T.

1. **Fact:** Let $\{x_1(t), x_2(t), \cdots, x_N(t)\}$ be a basis for V, then

$$y(t) = \sum_{i=1}^{N} a_i x_i(t) \qquad y(t) \leftrightarrow \mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix}^T$$

2. **Definition:** *Inner product* and *norm* of signals

$$\langle x_i(t), x_j(t) \rangle = \int_0^T x_i(t) x_j(t) dt$$

$$\|x_i(t)\| = \sqrt{\langle x_i(t), x_i(t) \rangle} = \sqrt{\int_0^T x_i(t) x_j(t) dt} = \sqrt{E_{x_i}}$$

3. **Fact:** $\{x_1(t), x_2(t), \cdots, x_N(t)\}$ are orthonormal means

$$\langle x_i(t), x_j(t) \rangle = \int_0^T x_i(t) x_j(t) dt = \begin{cases} E_{x_i} = 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

4. **Definition:** Unit energy signal

$$u(t) = \frac{y(t)}{\|y(t)\|} = \frac{y(t)}{\sqrt{E_y}} \qquad \|u(t)\| = \sqrt{E_u} = 1$$

5. **Representation:** $\{x_1(t), x_2(t), \cdots, x_N(t)\}$ is an orthonormal basis for V.

$$y(t) \in V \quad y(t) = \sum_{i=1}^{N} a_i x_i(t) \qquad a_i = \langle y(t), x_i(t) \rangle$$

$$y(t) \notin V \quad y(t) = y_s(t) + y_n(t)$$

$$y_s(t) = \sum_{i=1}^{N} a_i x_i(t) \qquad a_i = \langle y(t), x_i(t) \rangle$$

$$y_n(t) = y(t) - y_s(t) \qquad \langle y_s(t), y_n(t) \rangle = 0$$

6. **Fact:** Let $y(t) \leftrightarrow \mathbf{a}$ and $z(t) \leftrightarrow \mathbf{b}$.

$$E_y = \int_0^T y^2(t) dt = \|y(t)\|^2 = \|\mathbf{a}\|^2 = \sum_{i=1}^{N} a_i^2$$

$$\langle y(t), z(t) \rangle = \int_0^T y(t) z(t) dt = \sum_{i=1}^{N} a_i b_i = \langle \mathbf{a}, \mathbf{b} \rangle$$

$$d(y(t), z(t)) = \|y(t) - z(t)\| = \|\mathbf{a} - \mathbf{b}\| = d(\mathbf{a}, \mathbf{b})$$