Vector Space Representations of Signals

We will make use of the concept of a vector space in digital communications.

Review of Finite Dimensional Vector Spaces

Definition: An N-dimensional *vector space* is a set V of objects along with a definition of how to add two elements of V and a definition of how to scale an element by a real number. The example to keep in mind is the one you are already familiar with–three dimensional Euclidean space ($N = 3$).

Definition: *Linear Combination*

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N \qquad \text{or} \qquad \mathbf{y} = \sum_{i=1}^{N} a_i\mathbf{x}_i$$

Definition: A set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ of vectors in V is said to *span* V, if every vector $\mathbf{y}$ in V can be written as a linear combination of the elements in the set.

Definition: A set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ of vectors in V is called *linearly independent* if the equation

$$0 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N$$

holds only for $a_1 = a_2 = \cdots = a_N = 0$. Otherwise they are *linearly dependent*.

Definition: A set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ of vectors in V is called a *basis* for V if they span V and are linearly independent.

Theorem: If the set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\}$ is a basis for V then in the representation of $\mathbf{y}$

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_N\mathbf{x}_N$$

the coefficients $a_1, a_2, \cdots, a_N$ are unique. The numbers a_i are called the coordinates of $\mathbf{y}$ with respect to the given basis for V.

Definition: The *dimension* of a vector space V is the number of elements in a basis for V.

Definition: Let us construct the unique $N \times 1$ *coordinate vector* corresponding to the element $\mathbf{y}$ in V.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

Note that the coordinate vector "lives" in N-dimensional Euclidean space regardless of the nature of the vector space V. For example, V might be a vector space of polynomials. In digital communications, V is a space of functions.

Operations on Vectors in Euclidean Space

In this section, let V be N-dimensional Euclidean space and let $\mathbf{a}, \mathbf{b}$ be elements of V.

Definition: *Inner Product*

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{N} a_i b_i = \langle \mathbf{b}, \mathbf{a} \rangle$$

In one sense, the inner product measures how alike or parallel two vectors are.

Definition: *Norm* of a vector $\mathbf{a}$

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\sum_{i=1}^{N} a_i^2}$$

The norm measures the length of a vector. The norm satisfies the two inequalities

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$
$$\langle \mathbf{a}, \mathbf{b} \rangle \leq |\langle \mathbf{a}, \mathbf{b} \rangle| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|$$

Definition: *Angle* between two vectors $\mathbf{a}, \mathbf{b}$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cos\theta$$

Two vectors are *orthogonal* if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$.

Definition: *Unit Vector*

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \qquad\qquad \|\mathbf{u}\| = 1$$

Definition: *Distance* between two vectors $\mathbf{a}, \mathbf{b}$

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| = \sqrt{\sum_{i=1}^{N} (a_i - b_i)^2}$$

Definition: The set of vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N$ are called *orthonormal* if

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

Geometric Representation of Signals

All of the above ideas carry over to vector spaces of signals. We can define the length of a signal and the angle and distance between two signals. First we need to define a basis for our vector space of signals V. Let $x_1(t), x_2(t), \cdots, x_N(t)$ be a basis for our space. This means that they span the space and are linearly independent. So, any other signal $y(t)$ in the space can be represented as a linear combination

$$y(t) = \sum_{i=1}^{N} a_i x_i(t)$$

where the coordinates a_i are unique.

If we say that the basis is orthonormal then we know that

$$\langle x_i(t), x_j(t) \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

but how do we calculate the inner product $\langle x_i(t), x_j(t) \rangle$? There are many ways that this could be done. We will abide by the following definitions.

Definition: *Inner product* and *norm* of signals

$$\langle x_i(t), x_j(t) \rangle = \int_0^T x_i(t) x_j(t) dt$$
$$\|x_i(t)\| = \sqrt{\langle x_i(t), x_i(t) \rangle} = \sqrt{\int_0^T x_i(t) x_j(t) dt}$$

Note that the norm or length of a signal is just the square root of the energy of the signal.

$$\|x_i(t)\| = \sqrt{E_{x_i}}$$

Hence, the basis $\{x_i(t)\}$ is orthonormal if $x_i(t)$ has unit energy and is orthogonal to $x_j(t)$ for all $j \neq i$.

Let $y(t)$ be an arbitrary signal and let V be an N-dimensional signal space with $\{x_i(t)\}$ an orthonormal basis. Then $y(t)$ can be decomposed into two parts

$$y(t) = y_x(t) + y_n(t)$$

where $y_x(t)$ is the part of $y(t)$ "living" in V and $y_n(t)$ is orthogonal to V. We have the following relations.

$$y_x(t) = \sum_{i=1}^{N} \langle y(t), x_i(t) \rangle \; x_i(t) = \sum_{i=1}^{N} a_i x_i(t)$$
$$y_n(t) = y(t) - y_x(t)$$
$$\langle y_x(t), y_n(t) \rangle = 0$$

In digital communications, this idea will be used in the following way. The signal space will be spanned by a basis of orthonormal pulses. The transmitted signal will be a linear combination of the basis pulses. The received signal is equal to the transmitted signal plus noise. The noise signal has a component that lives in the signal space and an orthogonal component. The orthogonal component is removed by the receiver. Hence, the only part of the noise that may cause errors at the receiver is the noise component in the signal space.

Definition: Unit energy signal

$$u(t) = \frac{y(t)}{\|y(t)\|} = \frac{y(t)}{\sqrt{E_y}} \qquad \|u(t)\| = \sqrt{E_u} = 1$$

Once an orthonormal basis for the N-dimensional signal space is specified, any vector in the space is represented uniquely by the numbers

$$a_i = \langle y(t), x_i(t) \rangle \qquad y(t) = \sum_{i=1} a_i x_i(t)$$

Because of the unique relationship

$$y(t) \leftrightarrow \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

we can work with the vector $\mathbf{a}$ rather than the signal $y(t)$. This is useful because we can do all our analysis and design without having to worry about the waveforms. It will also make calculations easier and lead to some intuition that we would not get otherwise. The main requirement to utilize the relationship between the signal $y(t)$ and its vector representation $\mathbf{a}$ in Euclidean space is that the the elements of the vector $\mathbf{a}$ must be the coordinates of $y(t)$ with respect to an orthonormal basis for the signal space. That is why our text book goes through the Gram-Schmidt orthonormalization process.

Let $\mathbf{a}$ be the vector representation of the signal $y(t)$ and let $\mathbf{b}$ be the same for the signal $z(t)$. The following relationships hold.

$$E_y = \int_0^T y^2(t)dt = \|y(t)\|^2 = \|\mathbf{a}\|^2 = \sum_{i=1}^{N} a_i^2$$
$$\langle y(t), z(t) \rangle = \int_0^T y(t)z(t)dt = \sum_{i=1}^{N} a_i b_i = \langle \mathbf{a}, \mathbf{b} \rangle$$
$$d(y(t), z(t)) = \|y(t) - z(t)\| = \|\mathbf{a} - \mathbf{b}\| = d(\mathbf{a}, \mathbf{b})$$