1 Continuous-time Sampling

Let $x(t)$ be a continuous-time signal that is to be sampled at a rate $f_s = 1/T$ samples/second to produce the discrete-time sequence $x_n = x(t)|_{t=nT} = x(nT) = x(n/f_s)$. The sampling process may be modeled by modulating the signal to be sampled, i.e. $x(t)$ in this case, with a train of Dirac impulses followed by an operation that integrates the impulses to extract the areas. This illustrated in Figure 1.

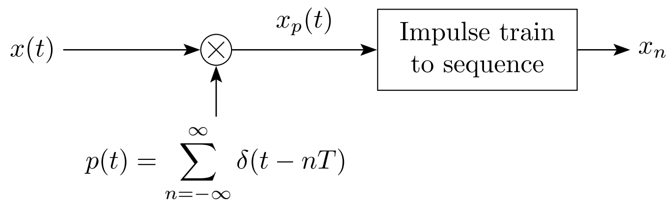

Figure 1: Model of the process of sampling a continuous-time signal to produce a discrete-time sequence.

Define the impulse train by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

The modulated signal is

$$x_p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT).$$

Note that the desired sample values $x(nT)$ weight the delta functions and may therefore be extracted by integration. Integrating over the interval $[iT - \frac{T}{2}, iT + \frac{T}{2})$ produces the i^{th} sample as follows,

$$\begin{aligned}
\int_{iT-\frac{T}{2}}^{iT+\frac{T}{2}} x_p(t)dt &= \int_{iT-\frac{T}{2}}^{iT+\frac{T}{2}} \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)dt \\
&= \sum_{n=-\infty}^{\infty} x(nT) \underbrace{\int_{iT-\frac{T}{2}}^{iT+\frac{T}{2}} \delta(t - nT)dt}_{\delta_{i-n} = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}} \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta_{i-n} \\
&= x(iT) = x_i.
\end{aligned}$$

The two signal processing operations involved in sampling a continuous-time signal, i.e. those illustrated in Figure 1, can be analyzed in the frequency domain. Let the continuous-time Fourier transforms (CTFT) of $x(t)$ and $x_p(t)$ be $X(\omega)$ and $X_p(\omega)$, respectively, and let the discrete-time Fourier transform of x_n be $X(e^{j\Omega})$. Suppose $x(t)$ has the Fourier transform shown in Figure 2.

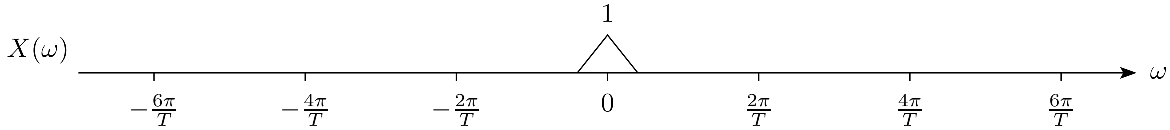

Figure 2: Fourier transform $X(\omega)$ of $x(t)$.

The CTFT of $p(t)$ is

$$p(t) \quad \longleftrightarrow \quad P(\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s), \quad \omega_s = \frac{2\pi}{T}.$$

This is illustrated in Figure 3 for frequencies from $-\frac{6\pi}{T}$ to $\frac{6\pi}{T}$. However, keep in mind that $P(\omega)$ is periodic with period $\frac{2\pi}{T}$ and therefore extends from $-\infty$ to ∞.

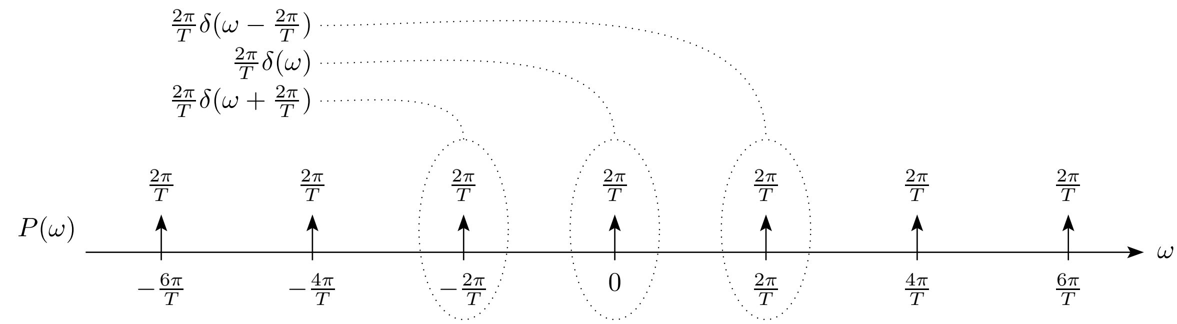

Figure 3: Fourier transform $P(\omega)$ of $p(t)$.

The modulation property of the Fourier transform leads to

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{1}$$

Graphically, this says that $X(\omega)$ is simply reproduced at integer multiples of the sample frequency $\omega_s = \frac{2\pi}{T}$. The amplitude of each replica is scaled by $\frac{1}{T}$. The CTFT of $x_p(t)$ is illustrated in Figure 4.

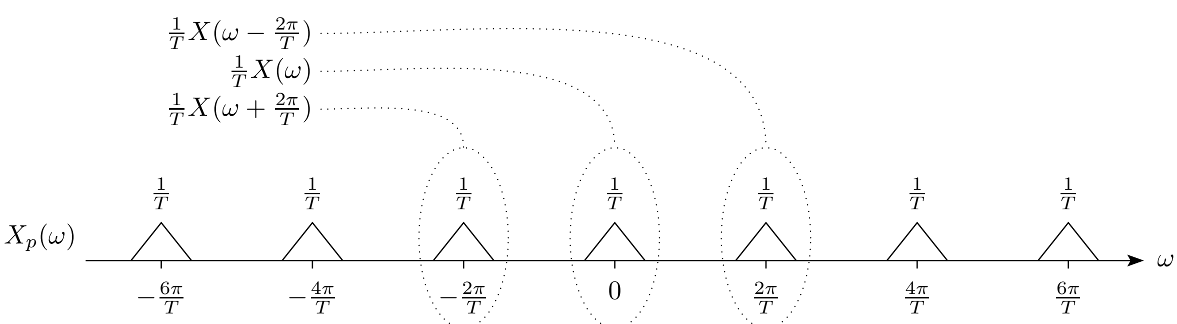

Figure 4: The Fourier transform $X_p(\omega)$ of $x_p(t)$ consists of scaled replicas of $X(\omega)$ appearing at multiples of the sample rate $\omega_s = \frac{2\pi}{T}$.

The relationship between CTFT of $x_p(t)$ and the DTFT of x_n can be derived as follows. First note that the DTFT of x_n is

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_n e^{-j\Omega n}. \tag{2}$$

Now, consider the CTFT of $x_p(t)$,

$$\begin{aligned} X_p(\omega) &= \int_{-\infty}^{\infty} x_p(t) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(t - nT) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega T n} \\ &= \sum_{n=-\infty}^{\infty} x_n e^{-j\omega T n}. \end{aligned} \tag{3}$$

Comparing (2) and (3) indicates that

$$X(e^{j\Omega}) = X_p\left(\frac{\Omega}{T}\right). \tag{4}$$

Now, substituting from (1), connects $X(e^{j\Omega})$ to $X(\omega)$ as follows,

$$X(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\Omega}{T} - \frac{2\pi}{T} k\right).$$

Graphically, this says that $X(\omega)$ is simply reproduced at integer multiples of the sample frequency ω_s, scaled by $\frac{1}{T}$, and then the frequencies are multiplied by T, i.e. $\Omega = \omega T$. The DTFT of x_n is illustrated in Figure 5.

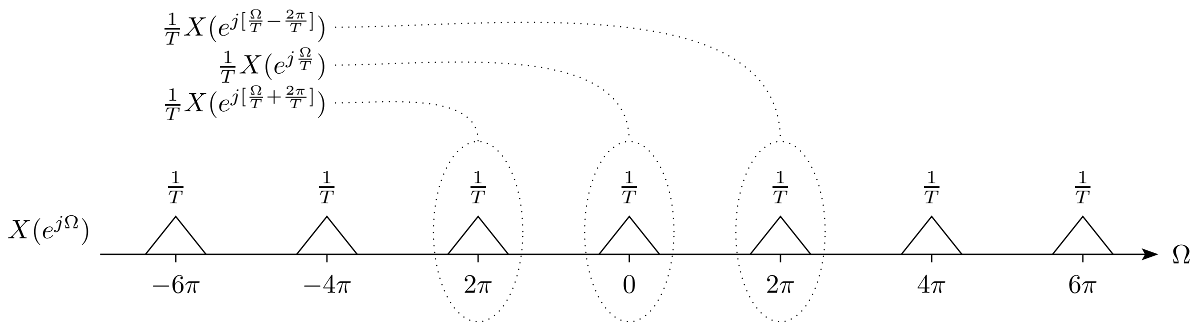

Figure 5: Fourier transform $X(e^{j\Omega})$ of x_n.

For convenience, Fourier transforms of all signals involved in the sampling process, i.e. those shown in Figure 1, are shown together in Figure 6. Note that the highest frequence ω_m in $X(\omega)$ gets mapped to $\Omega_m = \omega_m T$ through the sampling process.

In drawing Figure 6, the sample frequency was implicitly assumed to satisfy $\omega_s > 2\omega_m$. If this were not the case, the replicas of $X(\omega)$ in $X_p(\omega)$ would overlap and aliasing would result.

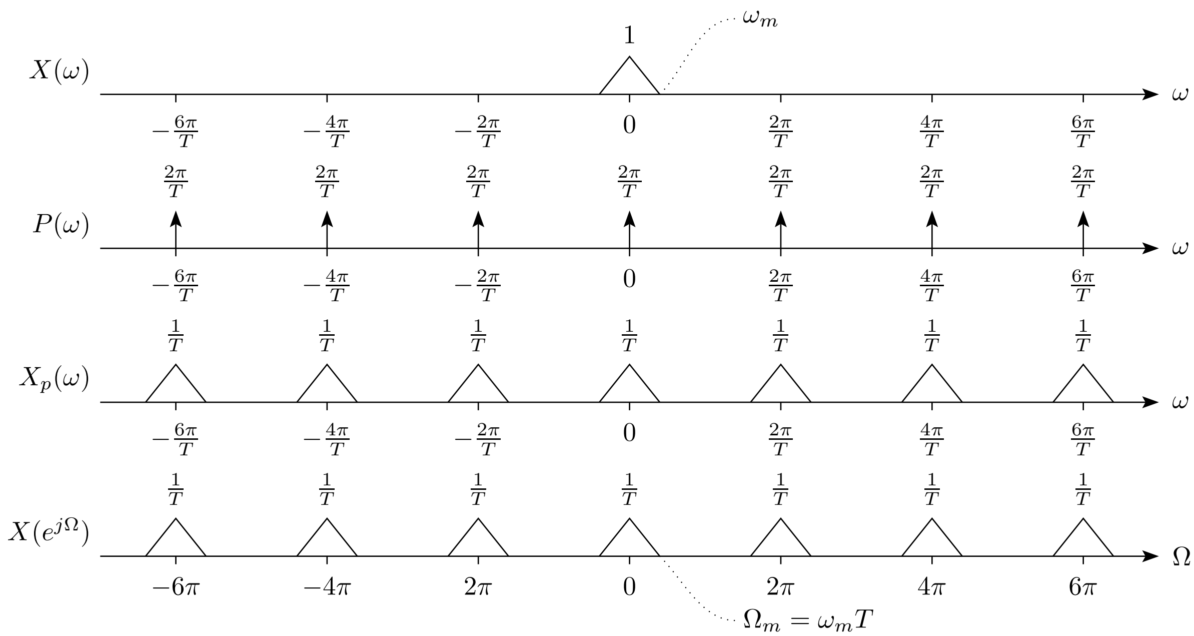

Figure 6: Fourier transforms of all signals involved in the continuous-time sampling process shown in Figure 1.

2 Continuous-time Reconstruction

Reconstructing $x(t)$ from samples x_n performs the operations illustrated in Figure 7

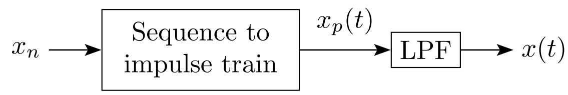

Figure 7: Model of the process of reconstructing a continuous-time signal from a discrete-time sequence.

First a weighted impulse train is created,

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_n \delta(t - nT).$$

In the frequency domain this corresponds to

$$X_p(\omega) = X(e^{j\omega T}), \tag{5}$$

which is the reverse of (4). Graphically, this says that the frequencies in $X(e^{j\Omega})$ are divided by T, i.e. $\omega = \frac{\Omega}{T}$. The conversion from $X(e^{j\Omega})$ to $X_p(\omega)$ is illustrated in Figure 8.

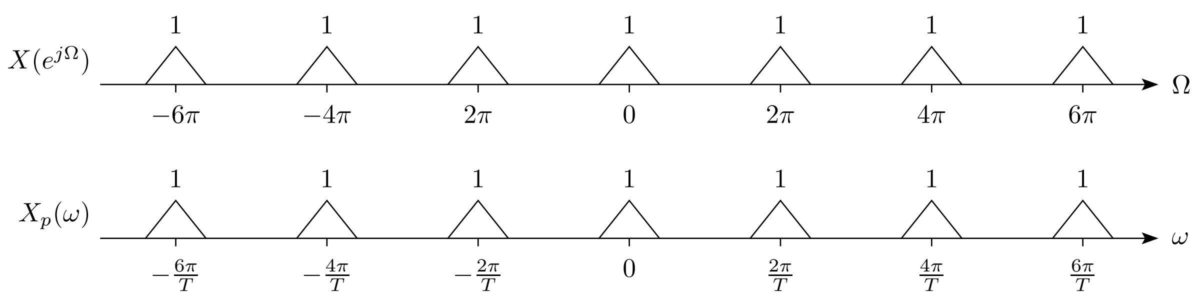

Figure 8: The DTFT of x_n and CTFT of $x_p(t)$ which are related by $X_p(\omega) = X(e^{j\omega T})$ through a scaling of the frequency axis, i.e. $\omega = \frac{\Omega}{T}$.

A low pass filter with impulse response $h(t)$ is used to remove the unwanted frequency components in $x_p(t)$. Suppose, an ideal low pass filter is used. Figure 9 shows the low pass filter frequency response $H(\omega)$. The cutoff frequency is half the sample rate, $\frac{\omega_s}{2} = \frac{\pi}{T}$. In the time domain, the action of the filter $h(t)$ may be viewed as interpolating between the values of $x_p(t)$ at the sample times $t = nT$.

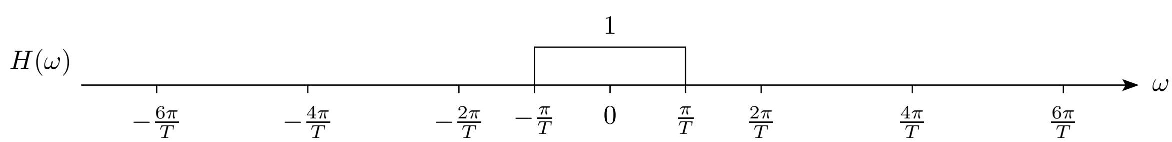

Figure 9: Frequency response of the low pass filter that removes the unwanted components of $X_p(\omega)$.

Filtering $x_p(t)$ by $h(t)$ leads to $x(t)$ with the Fourier transform given by,

$$X(\omega) = \begin{cases} X_p(\omega) & |\omega| < \frac{\omega_s}{2} = \frac{\pi}{T} \\ 0 & |\omega| > \frac{\omega_s}{2} = \frac{\pi}{T}. \end{cases}$$

Substituting from (5) leads to

$$X(\omega) = \begin{cases} X(e^{j\omega T}) & |\omega| < \frac{\omega_s}{2} = \frac{\pi}{T} \\ 0 & |\omega| > \frac{\omega_s}{2} = \frac{\pi}{T}. \end{cases}$$

The Fourier transform of the reconstructed continuous-time signal is shown in Figure 10 along with Fourier transform of the other signals involved in the reconstruction process, i.e. the signals shown in Figure 7.

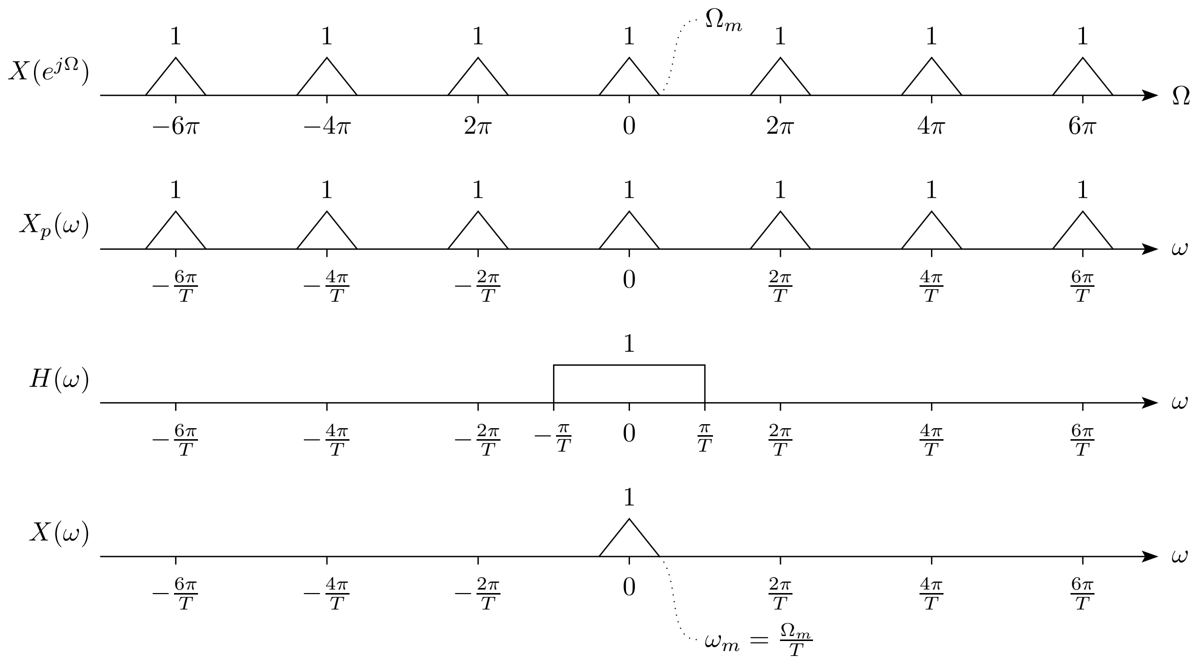

Figure 10: Fourier transforms of the signals involved in the reconstruction process shown in Figure 7.

3 Discrete-time Sampling: Down Sampling

Let x_n be a discrete-time sequence that is to be down sampled by a factor of N to produce the sequence $y_n = x_{nN}$. The down sampling process is modeled by modulation with a train of Kronecker impulses followed by an operation that removes the zeros introduced by modulation. This is illustrated in Figure 11.

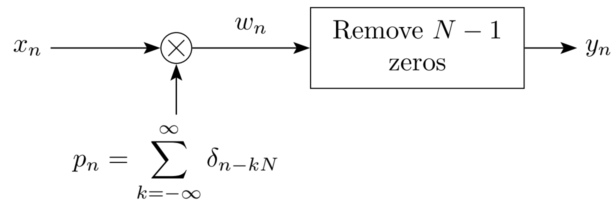

Figure 11: Model of the process of down sampling a discrete-time sequence.

Define the impulse train by

$$p_n = \sum_{k=-\infty}^{\infty} \delta_{n-kN}.$$

This is a sequence of zeros except for every N^{th} sample which is one. The modulated signal is

$$w_n = x_n \sum_{k=-\infty}^{\infty} \delta_{n-kN} = \sum_{k=-\infty}^{\infty} x_n \delta_{n-kN} = \sum_{k=-\infty}^{\infty} x_{kN} \delta_{n-kN}.$$

Note that the desired down sampled sequence x_{kN} weights the non-zero samples of w_n. The down sampled sequence is produced by removing the $N-1$ zeros in every N samples that are generated by modulation with p_n.

$$y_k = w_{kN}.$$

The two signal processing operations involved in down sampling can be analyzed in the frequency domain. Let the DTFTs of x_n, w_n, y_n be $X(e^{j\Omega}), W(e^{j\Omega}), Y(e^{j\Omega})$, respectively. The DTFT of p_n is

$$p_n \quad \longleftrightarrow \quad \Omega_s \sum_{m=-\infty}^{\infty} \delta(\Omega - m\Omega_s), \quad \Omega_s \sum_{k=0}^{N-1} \sum_{\ell=-\infty}^{\infty} \delta(\Omega - k\Omega_s - 2\pi\ell), \qquad \Omega_s = \frac{2\pi}{N}.$$

The DTFTs of x_n and p_n are illustrated in Figure 12. Note that in the plot of $P(e^{j\Omega})$, $N = 3$ was used.

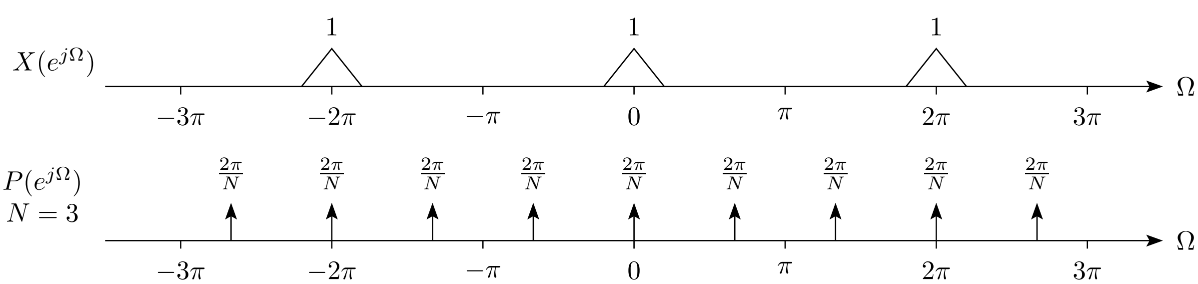

Figure 12: The DTFTs of x_n and p_n.

The modulation property of the DTFT involves circular convolution,

$$W(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \circledast P(e^{j\Omega}).$$

Applying this property leads to,

$$W(e^{j\Omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\Omega - k\Omega_s)}). \tag{6}$$

Graphically, this says that $X(e^{j\Omega})$ is simply reproduced at integer multiples of the sample frequency $\Omega_s = \frac{2\pi}{N}$. The amplitude of each replica is scaled by $\frac{1}{N}$. These operations are illustrated in Figure 13 for the case $N = 3$.

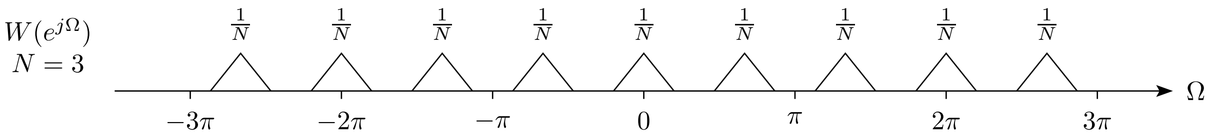

Figure 13: The DTFT of w_n contains N replicas of the DTFT of x_n which occur at integer multiples of the the sample frequency $\Omega_s = \frac{2\pi}{N}$.

The relationship between the DTFT of w_n and y_n can be derived as follows,

$$\begin{aligned}
Y(e^{j\Omega}) &= \sum_k y_k e^{-j\Omega k} \\
&= \sum_k w_{kN} e^{-j\Omega k} \\
&= \sum_{n=kN} w_n e^{-j\Omega n/N} \qquad \text{(change of variables in summation)} \\
&= \sum_{n=-\infty}^{\infty} w_n e^{-j\Omega n/N} \qquad \text{(because } w_n \text{ has } N-1 \text{ zeros between each nonzero sample)} \\
&= W(e^{j\Omega/N}).
\end{aligned} \tag{7}$$

Finally, substituting from (6) leads to

$$Y(e^{j\Omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\Omega - k2\pi)/N}\right). \tag{8}$$

Graphically, (7) says that $Y(e^{j\Omega})$ can be thought of as a frequency scaled version of $W(e^{j\Omega})$. Equation (8) says that $Y(e^{j\Omega})$ can be generated by reproducing $X(e^{j\Omega})$ at integer multiples of the sample frequency Ω_s, scaling the amplitude by $\frac{1}{N}$, and then multiplying all frequencies N, i.e. $\Omega \leftarrow \Omega N$. The DTFT of y_n is illustrated in Figure 14 along with the other signals involved in the down sampling process. Notice that the highest frequency Ω_0 in x_n is mapped to $\Omega_1 = \Omega_0 N$ in y_n through the down sampling process.

In drawing Figure 14, the sample frequency Ω_s was implicitly assumed to satisfy $\Omega_s = \frac{\pi}{N} > 2\Omega_0$. If this were not the case, the replicas of $X(e^{j\Omega})$ in $W(e^{j\Omega})$ would overlap and aliasing would result.

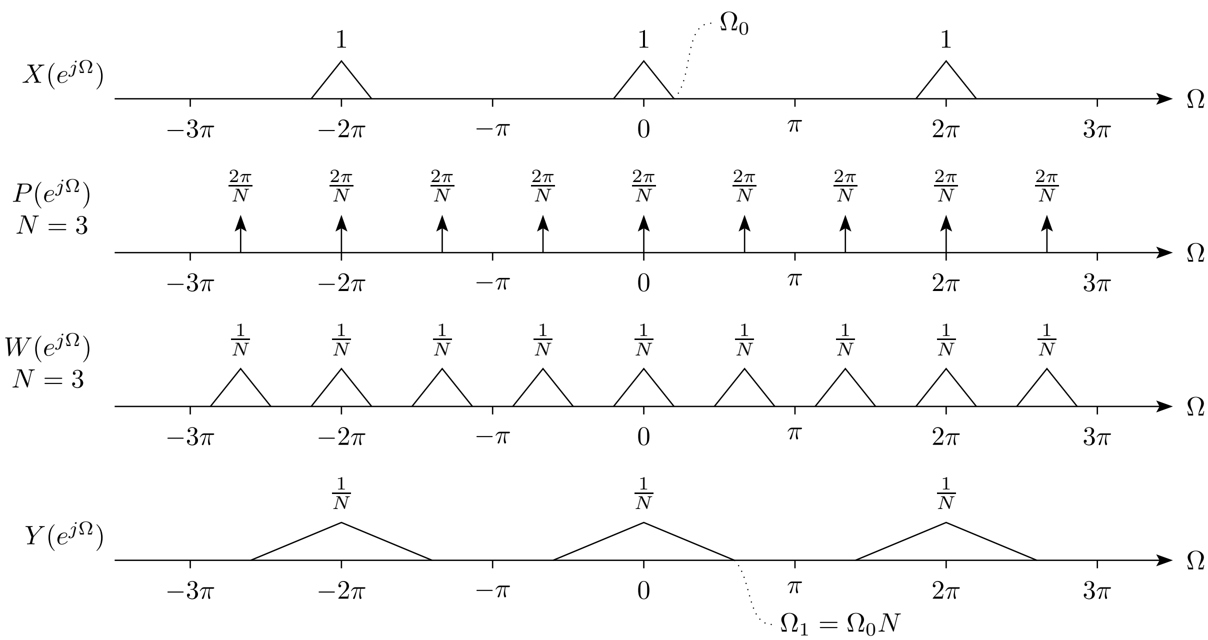

Figure 14: The DTFTs of the signals involved in the down sampling process.

4 Discrete-time Reconstruction: Up Sampling

Up sampling to obtain y_n from x_n reverses the operations in down sampling as illustrated in Figure 15.

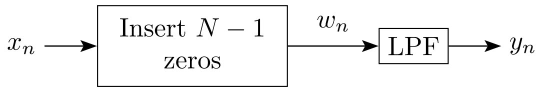

Figure 15: Model of the up sampling process.

First, w_n is obtained by inserting $N-1$ zeros in between each sample of x_k.

$$w_n = \begin{cases} x_{n/N} & n \text{ is a multiple of } N, \\ 0 & \text{otherwise.} \end{cases}$$

In the frequency domain this corresponds to

$$W(e^{j\Omega}) = X(e^{j\Omega N}). \tag{9}$$

Graphically, this says that the frequencies in $X(e^{j\Omega})$ are divided by N, i.e. $\Omega \to \frac{\Omega}{N}$. This creates N replicas of $X(e^{j\Omega})$. The desired replica is obtained by low pass filtering. The transition band edges are $\Omega_{\text{pass}} = \frac{\Omega_m}{N}$ and $\Omega_{\text{stop}} = \frac{\pi}{N} - \frac{\Omega_m}{N}$. The DTFT of the filter output y_n is

$$Y(e^{j\Omega}) = \begin{cases} W(e^{j\Omega}) & |\Omega| < \frac{\Omega_s}{2} = \frac{\pi}{N} \\ 0 & |\Omega| > \frac{\Omega_s}{2} = \frac{\pi}{N}. \end{cases}$$

Substituting from (9) leads to

$$Y(e^{j\Omega}) = \begin{cases} X(e^{j\Omega N}) & |\Omega| < \frac{\Omega_s}{2} = \frac{\pi}{N} \\ 0 & |\Omega| > \frac{\Omega_s}{2} = \frac{\pi}{N}. \end{cases}$$

The signals and systems involved in the up sampling process are illustrated in Figure 16 for the case $N = 3$.

Notice that the highest frequency Ω_0 in x_n gets mapped to $\Omega_1 = \frac{\Omega_0}{N}$ in the up sampled signal y_n.

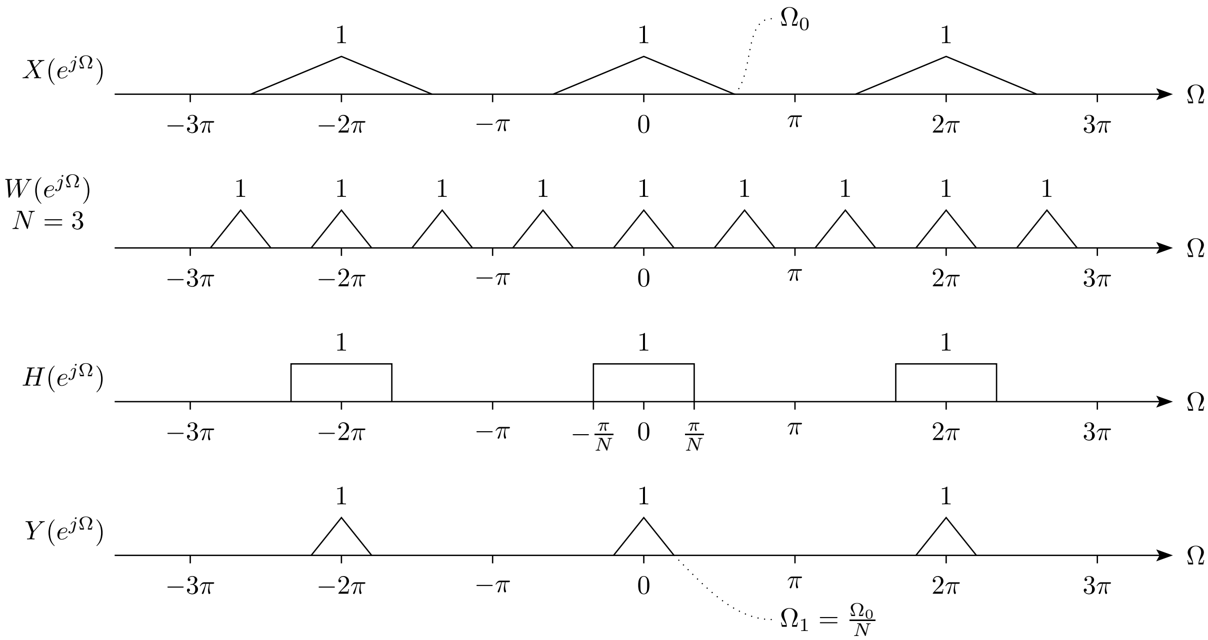

Figure 16: The DTFTs of the signals involved in up sampling.