Homework 1

Due: Wednesday, January 18, 2006

Reading

- Read the modulation chapter in the course reader.

Problems

1. Show that if $x(t)$ is real, then its Fourier transform, $X(j\omega)$, is Hermitian symmetric, $X^*(j\omega) = X(-j\omega)$.

2. Show that if $x(t)$ is real, then the real and imaginary parts of the Fourier transform, $\Re\{X(j\omega)\}$ and $\Im\{X(j\omega)\}$, are even and odd functions:

$$\Re\{X(j\omega)\} = \Re\{X(-j\omega)\},$$
$$\Im\{X(j\omega)\} = -\Im\{X(-j\omega)\}.$$

3. Derive the modulation property of the Fourier transform under the two definitions of the Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,$$
$$x(t) = \int_{-\infty}^{\infty} X(j2\pi f)e^{j2\pi f t}df \qquad X(j2\pi f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt.$$

4. Let $y(t)$ be a complex signal. Let $w(t) = \Re\{y(t)\}$ and $z(t) = j\Im\{y(t)\}$. Show that the Fourier transforms of $w(t), z(t)$, and $y(t)$ are related by:

$$W(j\omega) = \text{Herm}\{Y(j\omega)\},$$
$$Z(j\omega) = \text{AHerm}\{Y(j\omega)\},$$

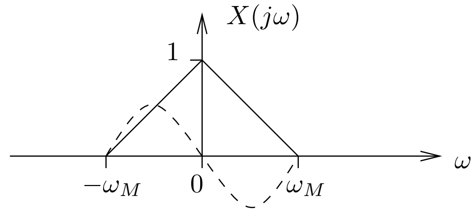
where $\text{Herm}\{z(t)\}$ and $\text{AHerm}\{z(t)\}$ are the Hermitian and anti-Hermitian part of of the complex number $z(t)$ and are computed as follows:

$$\text{Herm}\{z(t)\} = \frac{1}{2}[z(t) + z^*(-t)],$$
$$\text{AHerm}\{z(t)\} = \frac{1}{2}[z(t) - z^*(-t)].$$

5. Use Euler's identities $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ and $\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ to derive the following trigonometric identities:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)],$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)],$$
$$\cos(\alpha)\sin(\beta) = \frac{1}{2}[-\sin(\alpha-\beta) + \sin(\alpha+\beta)],$$
$$\cos^2(\theta) = \frac{1}{2}[1 + \cos(2\theta)],$$
$$\sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)],$$
$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta).$$

6. Let $x(t)$ be a real information signal with spectrum shown below. The real and imaginary parts of the Fourier transform $X(j\omega)$ are shown using solid and dashed lines respectively.



Define the following carrier signals.

$$c_1(t) = e^{j(\omega_c t + \theta_c)}$$
$$c_2(t) = e^{j\omega_c t}$$
$$c_3(t) = \cos(\omega_c t + \theta_c)$$
$$c_4(t) = \cos(\omega_c t)$$

Define the modulated signals $y_i(t) = x(t)c_i(t)$ for $i = 1, 2, 3, 4$.

(a) Assume $\omega_c = \omega_M$. Sketch the spectra of $y_1(t)$.

(b) Assume $\omega_c = \frac{1}{2}\omega_M$. Sketch the spectra of $y_2(t)$.

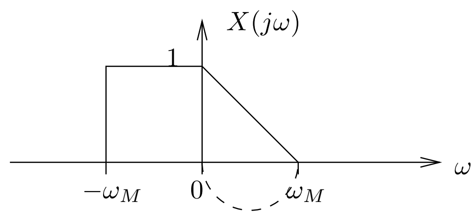
(c) Assume $\omega_c = 2\omega_M$. Sketch the spectra of $y_3(t)$.

(d) Assume $\omega_c = \frac{1}{2}\omega_M$. Sketch the spectra of $y_4(t)$.

(e) Draw a block diagram showing how to recover $x(t)$ from $y_1(t)$. Using algebra and trigonometry, show that the recovered signal is $x(t)$.

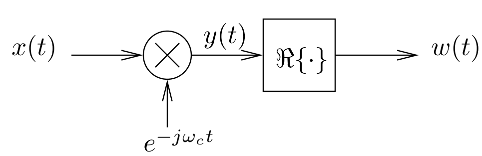
(f) Draw a block diagram showing how to recover $x(t)$ from $y_3(t)$. Using algebra and trigonometry, show that the recovered signal is $x(t)$.

7. Let $x(t)$ be the complex signal shown below.



The solid (dashed) line shows the real (imaginary) parts of $X(j\omega)$.

Consider the signal processing system shown below where $y(t)$ is a complex exponential modulation of $x(t)$ and $w(t)$ is the real part of $y(t)$. Assume that $\omega_c > \omega_M$.



(a) Using algebra and trigonometry, calculate expressions for $y(t)$ and $w(t)$.

(b) Using Fourier transform relations, calculate expressions for $Y(j\omega)$ and $W(j\omega)$.

(c) Sketch the spectra of $y(t)$ and $w(t)$.

(d) Draw a block diagram that shows how to recover $x(t)$ from $w(t)$. If any filters are used, indicate the critical frequencies.

(e) Mathematically show that the recovered signal is $x(t)$.

(f) Sketch the spectrum of each signal in your recovery system.

8. Using the results of problem 4 and the signal $x(t)$ given in problem 7, sketch the Fourier transform of $\Re\{x(t)\}$ and $j\Im\{x(t)\}$.

9. Suppose $x(t)$ has spectrum as shown in Problem 6 and is modulated by $c_1(t) = \cos(\omega_c t + \theta_1)$ followed by demodulation with $c_2(t) = \cos(\omega_c t + \theta_2)$ giving $y(t)$.

   (a) Derive a mathematical expression for $y(t)$ in terms of $x(t)$.

   (b) Sketch the spectrum of $y(t)$.

   (c) Describe is the effect on the recovered signal of a phase mismatch between the modulating and demodulating carrier signals?

   (d) What happens to $y(t)$ if $\theta 1 - \theta 2 = 27\pi/2$?

10. Suppose $x(t)$ has spectrum as shown in Problem 6 and is modulated by $c_1(t) = \cos(\omega_1 t + \theta_c)$ followed by demodulation with $c_2(t) = \cos(\omega_2 t + \theta_c)$ giving $y(t)$. Assume that $\omega_2 \approx \omega_1$.

    (a) Derive a mathematical expression for $y(t)$ in terms of $x(t)$.

    (b) Sketch the spectrum of $y(t)$.

    (c) Describe is the effect on the recovered signal of a frequency mismatch between the modulating and demodulating carrier signals?

11. Let $y(t)$ be a signal such that $Y(j\omega) = -4X(j[\omega + \omega_o])$. Suggest a signal $v(t)$ that can be used to recover $x(t)$ from $y(t)$ by multiplication, *i.e.*, $x(t) = v(t)y(t)$.