Hydraulic Pipe Transients by the Method of Characteristics

By Gilberto E. Urroz, September 2004

NOTE: The formulations shown in this document are taken from *Tullis, J.P.,* 1989 , *Hydraulics of Pipelines – Pumps, Valves, Cavitation, Transients*, John Wiley & Sons, New York

Hydraulic Transients

Hydraulic transients in pipelines occur when the steady-state conditions in a given point in the pipeline start changing with time, e.g., shutdown of a valve, failure of a pump, etc. In order to account for the disturbance in the steady-state conditions, a pressure wave will travel along the pipeline starting at the point of the disturbance and will be reflected back from the pipe boundaries (e.g., reservoirs) until a new steady-state is reached. The pressure waves in pipelines travel at a constant celerity as described next.

Celerity of pressure waves in pipelines

A pressure disturbance in a pipeline propagates with a very large wave celerity a given by

$$\boxed{a = \frac{\sqrt{\dfrac{K}{\rho}}}{\sqrt{1+\dfrac{C\cdot K\cdot D}{E\cdot \Delta D}}}},$$

where K and ρ are the bulk modulus of elasticity and density of the fluid, D and ΔD are the inner diameter and the thickness of the pipe, E is the Young modulus (modulus of elasticity) of the pipe material, and C is a coefficient that accounts for the pipe support conditions:

- $C = 1 - 0.5\mu$, if pipe is anchored at the upstream end only
- $C = 1-\mu^2$, if pipe is anchored against any axial movement
- $C = 1$, if each pipe section is anchored with expansion joints at each section

Here, μ represents the Poisson's ratio of the material.

For a perfectly rigid pipe, E is infinite, and the wave celerity simplifies to

$$\boxed{a = \sqrt{\frac{K}{\rho}}}.$$

Equations of hydraulic transients

The equations governing hydraulic transients are a pair of coupled, non-linear, first-order partial differential equations, namely,

$$g\frac{\partial H}{\partial x}+\frac{fV|V|}{2D}+V\frac{\partial V}{\partial x}+\frac{\partial V}{\partial t}=0 \qquad \text{(momentum)}$$

$$V\frac{\partial H}{\partial x}+\frac{\partial H}{\partial t}+\frac{a^2}{g}\frac{\partial V}{\partial x}=0, \qquad \text{(continuity)}$$

where H is the piezometric head ($=z+p/g$), V is the flow velocity, x is the distance along the pipe, t is time, g is the acceleration of gravity, f is the pipe friction factor (assumed constant), D is the pipe diameter, and a is the celerity of a pressure wave in the pipeline.

These equations can be simplified by recognizing that the advective terms $V\partial V/\partial x$ and $V\partial H/\partial x$, are negligible when compared to other terms in their corresponding equations. The resulting equations are, thus,

$$g\frac{\partial H}{\partial x}+\frac{fV|V|}{2D}+\frac{\partial V}{\partial t}=0 \qquad \text{(momentum)}$$

$$\frac{\partial H}{\partial t}+\frac{a^2}{g}\frac{\partial V}{\partial x}=0 \qquad \text{(continuity)}$$

The solution of the two transient equations require us to find the values of $H(x,t)$ and $V(x,t)$ in a domain defined by $0 < x < L$, and $0 < t < t_{max}$, where L is the length of the pipe, and t_{max} is an upper limit for the time domain. While the simultaneous solution of these two equations is possible through the use of finite differences, in this document we will approach the solution by determining the characteristic lines of the problem.

Characteristics in pipe transients

The characteristic lines for this case will be lines in the x-t plane defined by $dx/dt = u(x,t)$, along which a function of H and V is conserved in time. Since the momentum and continuity equations shown above include partial derivatives of H and V with respect to x and t, we will try to reconstruct out of them expressions for the total derivatives of H and V, namely,

$$dH/dt = (\partial H/\partial x)(dx/dt)+ \partial H/\partial t,$$

and

$$dV/dt = (\partial V/\partial x)(dx/dt)+ \partial V/\partial t.$$

We start by adding the momentum equation to the continuity equation multiplied by a parameter λ, i.e.,

$$g\frac{\partial H}{\partial x}+\frac{fV|V|}{2D}+\frac{\partial V}{\partial t}+\lambda\cdot\left(\frac{\partial H}{\partial t}+\frac{a^2}{g}\frac{\partial V}{\partial x}\right)=0.$$

Expanding this equation and collecting terms that resemble the total derivatives dH/dt and dV/dt, produces:

$$\lambda \cdot \left(\frac{\partial H}{\partial t} + \frac{g}{\lambda}\frac{\partial H}{\partial x} \right) + \left(\frac{\partial V}{\partial t} + \frac{\lambda a^2}{g}\frac{\partial V}{\partial x} \right) + \frac{fV\,|\,V\,|}{2D} = 0 \qquad \text{(combined)}$$

The terms between parentheses in this equation correspond to dH/dt and dV/dt, respectively, if we take $dx/dt = g/\lambda$ for dH/dt and $dx/dt = \lambda a^2/g$ for dV/dt. Since the two expressions for dx/dt must be the same, we have

$$\frac{\lambda a^2}{g} = \frac{g}{\lambda}, \text{ or } \lambda = \pm\frac{g}{a}.$$

Therefore, the characteristic lines are given by

$$\frac{dx}{dt} = \pm a\,.$$

Since a is a constant value for pipelines, the characteristic lines are straight lines with slopes $+a$ (referred to as a C+ line) and $-a$ (referred to as a C- line). The combined equation listed above, after some manipulation, reduces to the two characteristic equations:

$$\frac{dH}{dt} \pm \frac{a}{g}\frac{dV}{dt} \pm \frac{af}{2gD}V\,|\,V\,| = 0\,.$$

Most solutions are sought in terms of the pipeline discharge, $Q = VA$, rather than V, thus, the characteristic equations can be re-written as:

$$\frac{dH}{dt} + \frac{a}{gA}\frac{dQ}{dt} + \frac{af}{2gDA^2}Q\,|\,Q\,| = 0, \quad \text{along} \quad C+\!: \frac{dx}{dt} = +a\,,$$

and

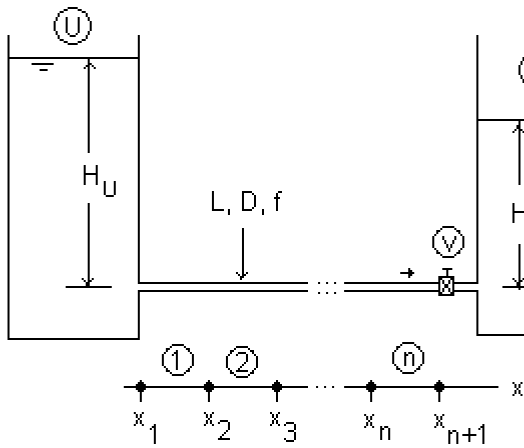
$$\frac{dH}{dt} - \frac{a}{gA}\frac{dQ}{dt} - \frac{af}{2gDA^2}Q\,|\,Q\,| = 0, \quad \text{along} \quad C-\!: \frac{dx}{dt} = -a\,.$$

Thus, the original systems of partial differential equations (momentum and continuity), has been reduced to the two *characteristic equations* (C+ and C-) along the corresponding characteristic lines, C+: $dx/dt = +a$, and C-: $dx/dt = -a$.

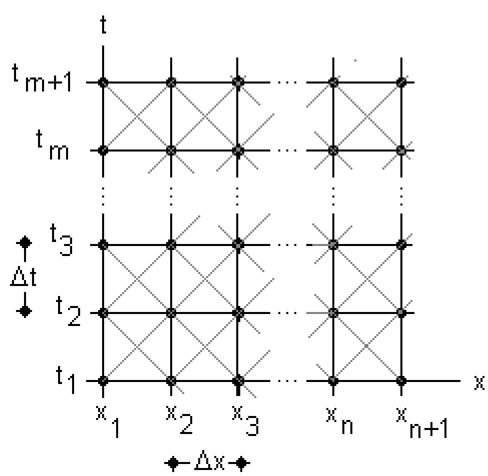
Numerical solution grid

A numerical solution of the transient problem in a pipeline will transport the values of H and Q along the characteristic lines as the time increases in increments Δt. Calculation nodes can be placed along the pipeline separated by increments Δx. The initial conditions of H and Q at $t = 0$, are then transported in time along the characteristic lines into the interior of the *x-t* solution domain. Boundary conditions at both extreme of the pipeline will be required to complete the solution. It is through these boundary conditions that a transient state is introduced into the system.

Consider the case represented by the following figure. In its steady state, water flows from reservoir U (upstream) to reservoir D (downstream) through a horizontal pipeline of length L, diameter D, and Darcy-Weisbach friction factor f (assumed constant). The water levels in both reservoirs are given as H_U and H_D. A valve V at the entrance to reservoir D will be closed to produce the transient in the pipeline. For the time being we consider only friction losses in the pipe and localized losses through the valve.



To proceed with the numerical solution we divide the pipe length into n sub-intervals of equal length Δx as illustrated in the figure above. The x solution domain starts at the pipe entrance $x_1 = 0$ and ends at the valve $x_{n+1} = L$. The solution is then calculated in the discretized domain shown in the following figure. Because we want the information from the nodes $x_1, x_2, \ldots, x_{n+1}$ to travel along the characteristic lines (i.e., along the diagonal lines in the figure above), we select the time interval Δt, such that $\boxed{a = \Delta x/\Delta t}$. Thus, the size of Δx and the wave celerity a will determine the size of our time interval.



The numerical solution consists in determining the values of

$$H_i^j = H(x_i, t_j) \text{ and } Q_i^j = Q(x_i, t_j)$$

at every grid point in the solution grid shown above. We selected n sub-intervals in x, such that $\boxed{\Delta x = L/n}$, and m sub-intervals in time such that $\boxed{t_{max} = t_1 + m\Delta t}$ (typically, $t_1 = 0$).

The conditions at the starting time $t_1 = 0$ (i.e., the *initial conditions*) are the steady state conditions before valve closure starts. Since the water level in reservoir U is constant, a boundary condition at $x=x_1$ (the pipe entrance) is that $H_1^j = H_U$ at any time level j. The boundary conditions at the valve V will be provided as a function of time. In other words, we will know how much the valve is being closed or the rate of closure and come up with a hydraulic equation to evaluate the head at the valve.

Initial conditions

The initial conditions are provided by the steady-state flow results. The energy equation written between the free-surfaces of the two reservoirs is

$$H_U = H_D + K_{vo}\frac{V_v^2}{2g} + f\frac{L}{D}\frac{V^2}{2g},$$

where K_{vo} is the valve loss coefficient for steady state conditions (a function of the valve opening). With $V_v = 4Q_o/A_v^2$, and $V = 4Q_o/A^2$, where A_v, A = valve and pipe cross-sectional areas, respectively, the solution for the steady-state discharge is

$$\boxed{Q_o = \sqrt{\frac{2g(H_U - H_D)}{\frac{K_v}{A_v^2} + \frac{fL}{DA^2}}} = Q_i^1}.$$

Thus, the piezometric head at the upstream end is $H_1^1 = H_U$ while that at the valve (upstream end) is

$$H_{n+1}^1 = H_D + K_v\frac{Q^2}{2gA_v^2}.$$

The piezometric head at any distance x from the upstream end of the pipe can be calculated by using:

$$H(x) = H_U - \frac{fQ^2}{2gDA^2}\cdot x.$$

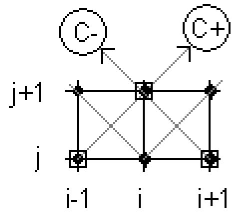
Therefore,

$$\boxed{H_i^1 = H_U - \frac{fQ^2}{2gDA^2}\cdot x_i}, \qquad \text{(H0)}$$

for $i = 1, 2, \ldots, n+1$.

Finite difference solution

Consider the sub-grid in the solution domain illustrated in the following figure:



We are interested in determining the values of H_i^{j+1} and Q_i^{j+1}, at interior point $(i,j+1)$, knowing the values of H and Q at points $(i-1,j)$ and $(i+1,j)$. The information from point $(i-1,j)$ will reach point (i,j) through characteristic line $C+$, while that from point $(i+1,j)$ will reach point (i,j) through characteristic line C-.

Consider first the solution along $C+: dx/dt = +a$. The corresponding combined equation is

$$\frac{dH}{dt}+\frac{a}{gA}\frac{dQ}{dt}+\frac{af}{2gDA^2}Q\,|\,Q\,|=0\,.$$

in which we make the following substitutions $B = a/(gA)$, $R = f/(2gDA^2)$, and $a = dx/dt$ (in the last term only). The resulting equation is:

$$\frac{dH}{dt}+B\frac{dQ}{dt}+RQ\,|\,Q\,|\frac{dx}{dt}=0\,.$$

Eliminating dt and integrating along the $C+$ line, i.e., from point $(i-1,j)$ to point (i,j) results in:

$$\int_{H_{i-1}^j}^{H_i^{j+1}} dH + B\int_{Q_{i-1}^j}^{Q_i^{j+1}} dQ + R\int_{x_{i-1}}^{x_i} Q\,|\,Q\,|\,dx = 0\,,$$

which, in turn, produces

$$H_i^{j+1}-H_{i-1}^j+B(Q_i^{j+1}-Q_{i-1}^j)+RQ_{i-1}^j\,|\,Q_{i-1}^j\,|\,(x_i-x_{i-1})=0\,.$$

Notice that, in order to evaluate the last integral in the equation above, it is necessary to know Q as a function of x along the characteristic $C+$. However, such information is not know a priori. The assumption made is that Q, along $C+$, will retain the value at point $(i,j-1)$. With that assumption, Q is taken out of the integral resulting in the last term in the last equation above.

Substituting $\Delta x = x_i - x_{i-1}$ in the equation above produces:

$$H_i^{j+1}-H_{i-1}^j+B(Q_i^{j+1}-Q_{i-1}^j)+RQ_{i-1}^j\,|\,Q_{i-1}^j\,|\,\Delta x=0\,. \qquad \text{(C+)}$$

A similar analysis along *C-: dx/dt = -a*, requires us to use the combined equation:

$$\frac{dH}{dt} - \frac{a}{gA}\frac{dQ}{dt} - \frac{af}{2gDA^2}Q\,|\,Q\,| = 0\,.$$

With $\boxed{B = a/(gA)}$, $\boxed{R = f/(2gDA^2)}$, and $a = -\,dx/dt$ (in the last term only), the combined equation becomes:

$$\frac{dH}{dt} - B\frac{dQ}{dt} + RQ\,|\,Q\,|\frac{dx}{dt} = 0\,.$$

Eliminating *dt* and integrating along the *C-* line, i.e., from point *(i+1,j)* to point *(i,j)* results in:

$$\int_{H_{i+1}^j}^{H_i^{j+1}} dH + B\int_{Q_{i+1}^j}^{Q_i^{j+1}} dQ + R\int_{x_{i+1}}^{x_i} Q\,|\,Q\,|\,dx = 0\,,$$

which, in turn, produces

$$H_i^{j+1} - H_{i+1}^j + B(Q_{i+1}^j - Q_1^{j+1}) + RQ_{i+1}^j\,|\,Q_{i-+1}^j\,|\,(x_i - x_{i+1}) = 0\,.$$

Here we use the assumption that *Q*, along *C-,* will retain the value at point (*i,j+1*).

Substituting $-\Delta x = x_i - x_{i+1}$ in the equation above produces:

$$H_i^{j+1} - H_{i+1}^j + B(Q_{i+1}^j - Q_1^{j+1}) - RQ_{i+1}^j\,|\,Q_{i-+1}^j\,|\,\Delta x = 0\,. \qquad \text{(C-)}$$

Since the values at points (*i-1,j*) and (*i+1,j*) are known, the following constants can be identified in equations (C+) and (C-):

$$\boxed{CP = H_{i-1}^j + BQ_{i-1}^j - R\cdot\Delta x\cdot Q_{i-1}^j\,|\,Q_{i-1}^j\,|,}$$

and

$$\boxed{CM = H_{i+1}^j - BQ_{i+1}^j + R\cdot\Delta x\cdot Q_{i+1}^j\,|\,Q_{i+1}^j\,|.}$$

With these constants, equations (C+) and (C-) become:

$$H_i^{j+1} = CP - B\cdot Q_i^{j+1}\,, \qquad \text{(C+)}$$

and

$$H_i^{j+1} = CM + B\cdot Q_i^{j+1}\,. \qquad \text{(C-).}$$

Notice that we have a system of two equations (C+,C-) in two unknowns Q_i^{j+1} and H_i^{j+1}. One way to solve the system is to write it as a matrix equation, $\mathbf{M}\cdot\xi=\eta$:

$$\begin{bmatrix} 1 & B \\ 1 & -B \end{bmatrix} \cdot \begin{bmatrix} H_i^{j+1} \\ Q_i^{j+1} \end{bmatrix} = \begin{bmatrix} CP \\ CM \end{bmatrix},$$

and solve for ξ, i.e., $\boxed{\xi = \mathbf{M}^{-1} \cdot \eta.}$ Or, $H_i^{j+1} = (CP+CM)/2,\ Q_i^{j+1} = (CP-CM)/(2B)$.

The solution of simultaneous solution of equations (C+) and (C-) can be carried on for all interior points in x in the solution domain, namely, for $i = 2, 3, \ldots, n$, and for time levels $j = 2, 3, \ldots, m$. The solution at points $i=1$ and $i = n+1$ for j = 2, 3, …, m require the use of the boundary conditions as detailed next.

Solution at $i = 1$

We indicated earlier that a boundary condition at $x=x_1$ (the pipe entrance) is that

$$\boxed{H_1^{j+1} = H_U} \qquad \text{(BC1)}$$

(a constant value) at any time level j. Since points at the upstream boundary condition are hit by the C- characteristic lines, at those points we have the following expression:

$$H_1^{j+1} = CM_1 + B \cdot Q_1^{j+1}, \qquad \text{(C1-)}$$

with

$$\boxed{CM_1 = H_2^j - BQ_2^j + R \cdot \Delta x \cdot Q_2^j \,|\, Q_2^j \,|.}$$

Solving (BC1) and (C1-) results in

$$\boxed{Q_1^{j+1} = \frac{H_U - CM_1}{B}}.$$

Solution at $i = n+1$

The pipeline under consideration includes a valve at its downstream end at point $x = x_{n+1} = L$. This valve will be closed either instantaneously or gradually to produce a transient in the pipeline. Thus, the boundary condition to consider at point $i = n+1$ is the energy loss equation for the valve.

Let H_{n+1}^{j+1} be the head just upstream of the valve and H_D the downstream reservoir elevation. The energy equation between the valve and the reservoir is written as

$$H_{n+1}^{j+1} = H_D + K_v \frac{V_{n+1}^{j+1} \,|\, V_{n+1}^{j+1} \,|}{2g},$$

where K_v is the valve loss coefficient. By using $V = 4Q/A_v^2$, where A_v is the valve area ($A_v = \pi D_v^2/4$), D_v = valve diameter, the energy equation becomes, with $\boxed{C_2 = K_v/(2gA_v^2)}$:

$$H_{n+1}^{j+1} = H_D + K_v \frac{Q_{n+1}^{j+1} \,|\, Q_{n+1}^{j+1} \,|}{2gA_v^2} = H_D + C_2 \cdot Q_{n+1}^{j+1} \,|\, Q_{n+1}^{j+1} \,|. \qquad \text{(BC2)}$$

Since point *(n+1,j+1)* is hit by characteristic *C+*, we also have the following equation available:

$$\boxed{H_{n+1}^{j+1} = CP_2 - B \cdot Q_{n+1}^{j+1}}, \quad \text{(C2+)}$$

with

$$\boxed{CP_2 = H_n^j + BQ_n^j - R \cdot \Delta x \cdot Q_n^j \mid Q_n^j \mid}$$

From equations (BC2) and (C2+) we get a quadratic equation:

$$\boxed{f(Q_{n+1}^{j+1}) = C_2 \cdot Q_{n+1}^{j+1} \mid Q_{n+1}^{j+1} \mid + B \cdot Q_{n+1}^{j+1} + H_D - CP_2 = 0}, \quad \text{(VEq)}$$

which can be solved numerically. With the solution for $Q_{n+1}{}^{j+1}$ from (VEq), equation (C2+) provides $H_{n+1}{}^{j+1}$. Notice that, once the valve is closed, you don't need to solve equation (VEq) anymore. Simply take $\boxed{Q_{n+1}{}^{j+1} = 0}$, and $\boxed{H_{n+1}{}^{j+1} = CP_2}$.

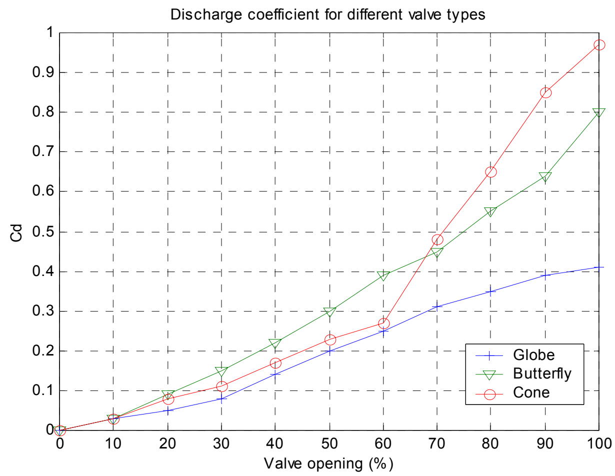
Valve loss coefficient

The valve loss coefficient K_v (used in the energy equations involving the valve) can be obtained from the following equation in terms of a discharge coefficient C_d :

$$\boxed{K_v = \frac{1}{C_d^2} - 1}. \quad \text{(VCEq)}$$

The discharge coefficient is, in turn, a function of the percentage of opening of the valve (percentage of open area to total open area), as shown in the table and figure below:

Value of C_d	Type of Valve		
% opening	Globe	Butterfly	Cone
0.0000	0.0000	0.0000	0.0000
10.0000	0.0300	0.0300	0.0300
20.0000	0.0500	0.0900	0.0800
30.0000	0.0800	0.1500	0.1100
40.0000	0.1400	0.2200	0.1700
50.0000	0.2000	0.3000	0.2300
60.0000	0.2500	0.3900	0.2700
70.0000	0.3100	0.4500	0.4800
80.0000	0.3500	0.5500	0.6500
90.0000	0.3900	0.6400	0.8500
100.0000	0.4100	0.8000	0.9700



(Data from Fig. 4.3, page 91, in Tullis, J.P., 1989 , *Hydraulics of Pipelines – Pumps, Valves, Cavitation, Transients*, John Wiley & Sons, New York)

Simulation of valve closing

The closing of the valve can be simulated by letting the percentage valve opening (VO) decrease from its initial setting, say VO_o (e.g., *50%, 75%*), to $VO = 0$, in the time interval $0<t<t_c$, where t_c = time to closure. Thus, $\boxed{VO(t) = VO_o(1-t/t_c)}$. In the simulation, therefore, we should be able to calculate $VO(t_j)$, and, using linear interpolation, determine the discharge coefficient, C_d, from the table above. The calculation of K_v follows from Equation (VCEq), shown above.

Data for simulation

For the purpose of running a solution, utilize the following values (use $D_v = D$):

Case	L (ft)	D (ft)	f	H_U (ft)	H_D (ft)	a (ft/s)	t_{max} (s)	t_c (s)	Valve type	Initial opening (%)
1	2000	1.00	0.000	50	0	1500	5	0	Cone	50%
2	2000	1.00	0.030	50	0	1500	50	0	Cone	50%
3	2000	1.00	0.030	50	0	1500	50	2.67	Cone	50%
4	17000	0.833	0.0123	750	200	1500	800	11.3	Butterfly	100%
5	17000	0.833	0.0123	750	200	1500	800	22.7	Butterfly	100%
6	17000	0.833	0.0123	750	200	1500	800	45.3	Butterfly	100%
7	17000	0.833	0.0123	750	200	1500	800	90.7	Butterfly	100%
8	17000	0.833	0.0123	750	200	1500	800	181	Butterfly	100%
9	17000	0.833	0.0123	750	200	1500	800	282	Butterfly	100%
10	17000	0.833	0.0123	750	200	1500	800	725	Butterfly	100%

In the table t_c stands for the time of closure, i.e., the time at which the valve should be fully closed assuming a linear closure function. Thus, at $t = 0$, valve opening = initial setting (%), and at $t = t_c$, valve opening = 0 %, in a linear fashion. The value $t_c = 0$ corresponds to an instant closure, i.e., the valve is open for the first time step only, and closed afterwards.

Algorithm outline

This is an outline of the algorithm to calculate the piezometric head H_i^j and the discharge Q_i^j for the pipeline transient problem described above:

1. Enter known parameters such as $L, D, D_v, f, H_U, H_D, a, t_{max}, t_c, VO_o, g, n$.
2. Calculate constant values such as A, A_v, K_{vo} (*), $Q_o, \Delta x, \Delta t, m,$ **M** (a matrix), x_i for $i = 1,2,..., n+1$, and t_j, for $j = 1,2,...,m+1$.

   (*) Note: Use an interpolation function (e.g., *interp1* with option '*linear*') to obtain the steady-state discharge coefficient, C_{do}, as a function of valve opening for the data in the table of page 9. Then, calculate K_{vo} from equation (VCEq) with $C_d = C_{do}$.

3. Create matrices $H_{ij} = H(x_i,t_j)$, and $Q_{ij} = Q(x_i,t_j)$, and initialize them to zero.
4. Load the initial conditions $Q_{i,1} = Q_o$, and $H_{i,1}$ from equation (H0).
5. Loop on time steps, i.e., for $j = 1, 2, ..., m$
   a. Calculate upstream boundary values $Q_1^{\ j+1}$ and $H_1^{\ j+1}$ (see <u>*Solution at i = 1*</u> in page 8)
   b. Calculate downstream boundary values $Q_{n+1}^{\ j+1}$ and $H_{n+1}^{\ j+1}$ (see <u>*Solution at i = n+1*</u> in pages 8 and 9).
      i. If $t_j<t_c$ (valve in process of closing),
         1. Use $VO(t) = VO_o(1-t/t_c)$ to calculate valve opening.
         2. Use an interpolation function (e.g., *interp1* with option '*linear*') to obtain C_d as a function of valve opening, *VO*, for the data in the table of page 9.
         3. Calculate K_v from equation (VCEq).
         4. Use function *fzero* to solve equation (Veq) at point x_{n+1}.
         5. Calculate $H_{n+1}^{\ j+1}$ from equation (C2+)
      ii. If $t_j \geq t_c$ (valve already closed),
         1. Make $Q_{n+1}^{\ j+1} = 0$, and $H_{n+1}^{\ j+1} = CP_2$.
   c. Calculate Q_i^{j+1} and H_i^{j+1} at the inner points, i.e., $i = 2, 3, ..., n$, using the matrix equation in page 8. Use left division or inverse matrices to solve the matrix equation.
6. Produce graphics using the results found in 5.

Exercise

(a) Write a Matlab function to calculate the piezometric head *H* and the discharge *Q* for the problem outlined in this document and for the data shown in the table above. Follow the algorithm outlined above.
(b) Write a scrip to run the function developed in (a) for cases 1 through 3 from the table above. Select your Δx so that you get 10 solution points along the pipeline.
(c) As part of the script developed in (b), produce an animation showing the variation of *H* vs. *x* for different time steps.

(d) Repeat the animation produced in (c) for the discharge *Q* vs. *x* for different time steps.
(e) For the data of cases 1 through 3, plot the variation of *H* vs. *t*, at three points along the pipeline: (1) the upstream boundary (*i = 1)*, (2) the mid point (*i = round(n/2)*), and the upstream boundary (*i = n+1*). Plot the three signals in the same plot, identifying the different signals with the command *legend*.
(f) Plot the variation of *Q* vs. *t* for the same points listed in (e). Use a single plot, identifying the different signals with the command *legend*.
(g) Prepare a script to run the solution developed in (a) for cases 4 through 10 in the table above, and obtain the maximum head at the valve (H_v^{max}) for each case. Then, plot H_v^{max} vs. t_c.

Additional information on hydraulic transients
Hydraulic transients are problems of practical applications in many hydraulic systems. The case presented here is a relatively simple case of a single pipeline, two reservoirs and a valve. More complicated cases may include two or more pipelines, branching pipelines, pipeline networks, free-flowing valves, pump failure, surge tanks, governed turbines, valve stroking, column separation (cavitation), oscillatory flow, etc.

If interested in additional information about hydraulic transients, consult the following references:

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