Solution to potential flow in rectangular domain

Implicit method with Dirichlet and Neumann boundary conditions

by Gilberto E. Urroz, November 2003 – modified October 2004

Part 1 – Fluid Mechanics Background

Velocity potential for inviscid incompressible fluid flow

The flow of an inviscid (or ideal) incompressible fluid is also known as potential flow, because there exists a function $\phi(x, y, z)$, known as the velocity potential that allows the calculation of the velocity components as follows:

$$\mathrm{u}(x, y, z) = \frac{\partial}{\partial x}\phi,\ \mathrm{v}(x, y, z) = \frac{\partial}{\partial y}\phi,\ \mathrm{w}(x, y, z) = \frac{\partial}{\partial z}\phi\ .$$

In other words, the velocity vector, $\mathbf{q}(x, y, z) = \mathrm{u}(x, y, z)\,\mathbf{i} + \mathrm{v}(x, y, z)\,\mathbf{j} + \mathrm{w}(x, y, z)\,\mathbf{k}$, is the *gradient* of the velocity potential, or

$$\mathbf{q} = grad\ \phi = \nabla\phi.$$

For an incompressible flow, the continuity equation is written as

$$div\ \mathbf{q} = \nabla\bullet\mathbf{q} = 0.$$

Where the operation $\nabla\bullet\mathbf{q}$ is known as the divergence of the velocity vector **q**. Thus, continuity is

$$\left(\frac{\partial}{\partial x}u\right) + \left(\frac{\partial}{\partial y}v\right) + \left(\frac{\partial}{\partial z}w\right) = 0\ .$$

With the velocity components defined above, the continuity equation becomes

$$\left(\frac{\partial^2}{\partial x^2}\phi\right) + \left(\frac{\partial^2}{\partial y^2}\phi\right) + \left(\frac{\partial^2}{\partial z^2}\phi\right) = 0\ .$$

Using vector analysis notation, we can write the latter result as $div\ (grad\ \phi) = \nabla\times(\nabla\phi) = 0$, or $\nabla^2\phi = 0$. The velocity potential, therefore, satisfies Laplace's equation. For a two-dimensional flow, the corresponding equation is

$$\left(\frac{\partial^2}{\partial x^2}\phi\right) + \left(\frac{\partial^2}{\partial y^2}\phi\right) = 0\ .$$

Streamfunction for two-dimensional flow

Consider the continuity equation for a two-dimensional incompressible flow:

$$\left(\frac{\partial}{\partial x}u\right)+\left(\frac{\partial}{\partial y}v\right)=0\ .$$

It is possible to define a function $\psi(x,y)$, known as the *streamfunction*, such that the velocity components are calculated as

$$u=\frac{\partial}{\partial y}\psi \text{ and } v=-\left(\frac{\partial}{\partial x}\psi\right).$$

These velocity components satisfy continuity, i.e.,

$$\left(\frac{\partial^2}{\partial x\,\partial y}\psi\right)+\left(\frac{\partial}{\partial y}\left(-\left(\frac{\partial}{\partial x}\psi\right)\right)\right)=\left(\frac{\partial^2}{\partial x\,\partial y}\psi\right)-\left(\frac{\partial^2}{\partial y\,\partial x}\psi\right)=0.$$

Streamfunction for two-dimensional irrotational flow

A measure of the rotationality of a flow is the quantity known as the *vorticity* of the flow, i.e.,

$$\zeta = curl\ \mathbf{q} = \nabla\times\mathbf{q}$$

where the operation $\nabla\times\mathbf{q}$ is known as the curl or rotational of the velocity vector **q**. If the vorticity of a flow is zero everywhere ($\zeta=0$), the flow is said to be *irrotational.* For a two-dimensional flow, the velocity is given by

$$\mathbf{q} = u(x,y,z)\mathbf{i} + v(x,y,z)\,\mathbf{j} + 0\mathbf{k} = u\mathbf{i} + v\mathbf{j},$$

and the vorticity is $\zeta=\left[\left(\frac{\partial}{\partial x}v\right)-\left(\frac{\partial}{\partial y}u\right)\right]\mathbf{k}$, i.e., the only non-zero component of vorticity is

$$\zeta_z=\left(\frac{\partial}{\partial x}v\right)-\left(\frac{\partial}{\partial y}u\right).$$

If the two-dimensional flow is irrotational, then $\zeta_z=0$, i.e.,

$$\left(\frac{\partial}{\partial x}v\right)-\left(\frac{\partial}{\partial y}u\right)=0.$$

Replacing the velocity components calculated in terms of the streamfunction, we find that this function also satisfies Laplace's equation, i.e.,

$$\left(\frac{\partial}{\partial x}\left(-\left(\frac{\partial}{\partial x}\psi\right)\right)\right)-\left(\frac{\partial^2}{\partial y^2}\psi\right)=0,$$

or

$$\left(\frac{\partial^2}{\partial x^2}\psi\right)+\left(\frac{\partial^2}{\partial y^2}\psi\right)=0\ .$$

Two-dimensional irrotational flows of incompressible fluids are also known as potential flows.

Equipotential lines

In a given two-dimensional flow, the lines described by $\phi(x,y) = constant$ are referred to as *equipotential lines* of the flow. The flow velocity **q** at any point of a equipotential line is perpendicular to the line, and flow occurs in the direction of decreasing values of $\phi(x,y)$.

Streamlines

The lines described by $\psi(x,y) = constant$ are referred to as the *streamlines* of the flow. The flow velocity **q** at any point of a streamline is tangent to the line, i.e., there is no flow across streamline. The flow occurs along the directions of the streamlines. Also, the unit discharge (discharge per unit width normal to the plane of the flow) between any two streamlines is given by the difference in the values of the streamlines.

Equipotential lines and streamlines – the flow net

The definition of the velocity potential and streamfunction in a two-dimensional potential flow is such that equipotential lines and streamlines cross each other at 90 degrees. Thus, the collection of equipotential lines and streamlines form a grid of lines known as the flow grid.

Flow velocities

The flow velocities for a two-dimensional potential flow are calculated by using

$$u = \frac{\partial}{\partial x}\phi = \frac{\partial}{\partial y}\psi \quad \text{and} \quad v = \frac{\partial}{\partial y}\phi = -\left(\frac{\partial}{\partial x}\psi\right).$$

Boundary conditions

Typically, cross-sections of the flow with constant velocity correspond to lines $\phi(x,y) = constant$, while solid boundaries require that no flow occurs normal to them, i.e., $\frac{\partial}{\partial n}\phi = 0$, where n indicates the direction normal to the boundaries. In terms of streamfunctions, it is typical to take $\psi(x,y) = 0$ at a solid boundary of a flow and $\psi(x,y) = Q$, the unit discharge, at the other solid boundary. Sections of the flow with constant velocity correspond to boundary conditions of the form $\frac{\partial}{\partial n}\psi = 0$, where n here represents a direction normal to the velocities at the section.

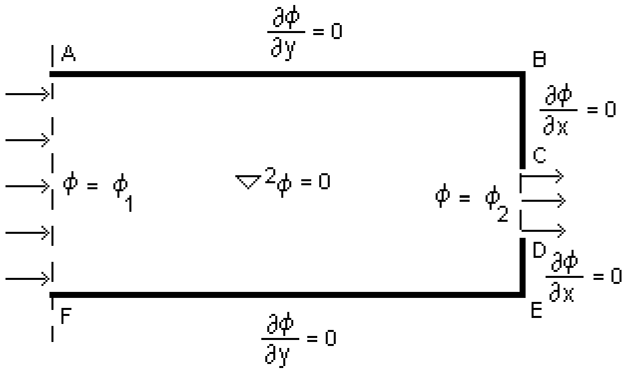
Groundwater flow

So far we have described the two-dimensional flow of an irrotational (vorticity = 0), inviscid (viscosity = 0), incompressible fluid (a liquid, or a gas undergoing negligible compression). Whereas all fluids are viscous, some flows are so nearly irrotational (except near solid walls where a *viscous boundary layer* develops) that they can be treated as potential flow. One such case is the case of groundwater flow. In this case, the velocity potential is defined by $\phi = z + \frac{p}{\gamma}$, where z is the elevation of a point measured from an arbitrary horizontal reference level (datum), p is the pressure in the soil pores, and γ is the specific weight of water (γ = 9806 N/m^3 = 62.4 lb/ft^3). The velocity components, $u = \frac{\partial}{\partial x}\phi$ and $v = \frac{\partial}{\partial y}\phi$, represent the so-called *Darcy velocities*, whereas the actual pore velocities are given by $u_P = \frac{u}{\nu}$ and $v_P = \frac{v}{\nu}$, where ν is the porosity of the aquifer of interest.

Part 2 – Mathematical Description

Mathematical description of the velocity potential problem

To illustrate the solution of Laplace's equation in terms of the velocity potential, $\phi(x, y)$, for a two-dimensional potential flow, consider the figure below. The thick lines represent solid walls and the thin lines represent openings in the rectangular domain of solution. The governing equation is Laplace's equation, namely, $\nabla^2\phi(x,y) = 0$.

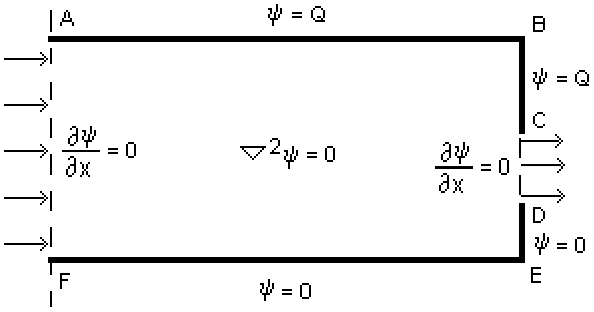


Along cross-sections AF and CD, where the velocity is horizontal and constant, we use the boundary conditions $\phi = \phi_1$ and $\phi = \phi_2$, respectively. Along solid boundaries ABC and FED, the velocity normal to the boundaries must be zero since there is no flow allowed across these boundaries. The corresponding zero normal velocities on those boundaries are shown as either $u = \frac{\partial}{\partial x}\phi = 0$ or $v = \frac{\partial}{\partial y}\phi = 0$ in the figure.

Mathematical description of the streamfunction problem

The following figure shows the governing equation, $\nabla^2\psi(x,y) = 0.$, and the boundary conditions for the same flow shown above, but this time expressed in terms of the streamfunction of the flow, $\psi(x, y)$.

Along the lines AF and CD the flow is horizontal, therefore, the vertical velocity component $v = -\left(\frac{\partial}{\partial x}\psi\right)$ = 0. The solid boundary FED corresponds to a streamline of flow since, for ideal flow, a flow particle entering the domain at F will be transported along FED. We select streamline FED to correspond to $\psi(x, y) = 0$. The solid boundary ABC is also a streamline of the flow corresponding to the value of the unit discharge, Q, i.e., $\psi(x, y) = Q$ along ABC.



Part 3 – Numerical Solution

Discretization of the solution domain
The solution domain is rectangular having a width *w* and a height *h*. The opening in the downstream boundary, namely, section CD, has a length h_0. The domain will be provided with a solution grid consisting of *n* equally-spaced points in the x-direction, i.e., line AF corresponds to *i = 1* or $x_1 = 0$, while line BCDE corresponds to *i = n* or $x_n = w$. The increment in the x-direction is given by $Dx = \frac{w}{n-1}$, from which it follows that $n = \text{floor}\left(\frac{w}{Dx}\right) + 1$. The values of x_i are calculated as $x_i = 0 + (i-1)\,Dx$. Similarly, in the y-direction we will have *m* equally-spaced points with *j = 1* (or $y_1 = -\frac{h}{2}$) along line FE and *j = m* (or $y_m = \frac{h}{2}$) along line AB. The increment in the y-direction is given by $Dy = \frac{h}{m-1}$, from which it follows that $m = \text{floor}\left(\frac{h}{Dy}\right) + 1$. The values of y_j are calculated as $y_j = -\frac{h}{2} + (j-1)\,Dy$. In order to locate points D and C in the downstream boundary, we associate them with indices j_1 and j_2 respectively, i.e., $y_D = y_{j_1}$ and $y_C = y_{j_2}$, with $j_1 < j_2$. If the opening CD in the downstream boundary is centrally located, the indices j_1 and j_2 are calculated as $j_1 = \text{floor}\left(\frac{h-h_0}{2\,Dy}\right) + 1$ and $j_2 = j_1 + \text{floor}\left(\frac{h_0}{Dy}\right)$. If the opening CD in the downstream boundary is not centrally located, the user must select the value of the index j_1 between 2 and *m-1* and calculate j_2 as shown above.

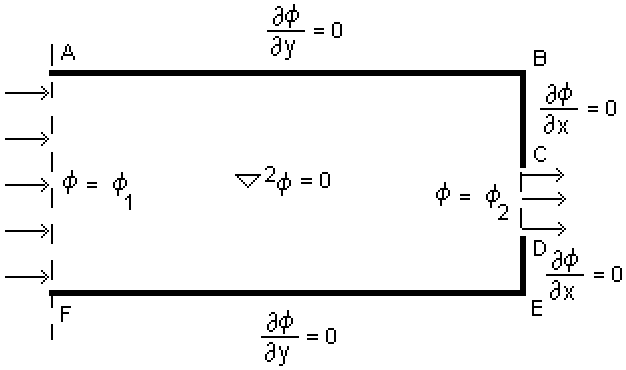
Numerical solution for the velocity potential
Consider first the solution of the Laplace equation for the velocity potential $\phi(x,y)$. The finite-difference approximation for this equation in the interior points of the solution grid (namely, for *i = 1, 2, ..., n-1* and *j = 1, 2, ..., m-1*) is given by the equations

$$\phi_{i+1,j} + \beta^2\,\phi_{i,j+1} + \phi_{i-1,j} + \beta^2\,\phi_{i,j-1} - 2\,(1+\beta^2)\,\phi_{i,j} = 0 \qquad [1]$$

where $\beta = \frac{Dx}{Dy}$. With this expression we can generate *(n-2)(m-2)* equations. To complete the solution we need to load the boundary conditions described in the diagram below.

For the Dirichlet-type boundary conditions in AF and CD we can write:

* $\phi_{1,j} = \phi_1$, for *j = 1, 2, ..., m* (along FA)
* $\phi_{n,j} = \phi_2$, for *j* = $j_1 + 1$, $j_1 + 2$, ..., $j_2 - 1$ (along DC)



For the Neumann-type boundary conditions we need to approximate the derivatives $\frac{\partial}{\partial x}\phi$ and $\frac{\partial}{\partial y}\phi$ with the following finite-differences:

*Along FE, for $\frac{\partial}{\partial y}\phi = 0$, we use $\frac{\phi_{i,2} - \phi_{i,0}}{2\,Dy} = 0$, i.e., $\phi_{i,0} = \phi_{i,2}$. Although the values $\phi_{i,0}$ are off the grid, we can replace them into the finite difference approximation for *j = 1*, namely, $\phi_{i+1,1} + \beta^2\,\phi_{i,2} + \phi_{i-1,1} + \beta^2\,\phi_{i,0} - 2\,(1 + \beta^2)\,\phi_{i,1} = 0$, to produce the set of equations:

$$\phi_{i+1,1} + 2\,\beta^2\,\phi_{i,2} + \phi_{i-1,1} - 2\,(1 + \beta^2)\,\phi_{i,1} = 0\,, \qquad [2]$$

for *i = 1, 2, ..., n-1.*

* Along AB, for $\frac{\partial}{\partial y}\phi = 0$, we use $\frac{\phi_{i,m+1} - \phi_{i,m} - 1}{2\,Dy} = 0$, i.e., $\phi_{i,m+1} = \phi_{i,m-1}$. Although the values $\phi_{i,m+1}$ are off the grid, we can replace them into the finite difference approximation for *j = m*, namely, $\phi_{i+1,m} + \beta^2\,\phi_{i,m+1} + \phi_{i-1,m} + \beta^2\,\phi_{i,m-1} - 2\,(1 + \beta^2)\,\phi_{i,m} = 0$, to produce the set of equations:

$$\phi_{i+1,m} + \phi_{i-1,m} + 2\,\beta^2\,\phi_{i,m-1} - 2\,(1 + \beta^2)\,\phi_{i,m} = 0\,, \qquad [3]$$

for *i = 1, 2, ..., n-1.*

* Along BC and DE, for $\frac{\partial}{\partial x}\phi = 0$, we use $\frac{\phi_{n+1,j} - \phi_{n-1,j}}{2\,Dx} = 0$, i.e., $\phi_{n+1,j} = \phi_{n-1,j}$. Although the values $\phi_{n+1,j}$ are off the grid, we can replace them into the finite difference approximation for *i = n*, namely, $\phi_{n+1,j} + \beta^2\,\phi_{n,j+1} + \phi_{n-1,j} + \beta^2\,\phi_{n,j-1} - 2\,(1+\beta^2)\,\phi_{n,j} = 0$, to produce the set of equations:

$$\beta^2\,\phi_{n,j+1} + 2\,\phi_{n-1,j} + \beta^2\,\phi_{n,j-1} - 2\,(1+\beta^2)\,\phi_{n,j} = 0\,, \qquad [4]$$

for *j = 2,3,...,*j_1 and *j =* $j_2\,, j_2+1\,, ..., m-1$

* At point B (*i = n, j = m*) we have both $\frac{\partial}{\partial x}\phi = 0$ and $\frac{\partial}{\partial y}\phi = 0$, i.e., $\frac{\phi_{n+1,m} - \phi_{n-1,m}}{2\,Dx} = 0$, or $\phi_{n+1,m} = \phi_{n-1,m}$, and $\frac{\phi_{n,m+1} - \phi_{n,m-1}}{2\,Dy} = 0$, or $\phi_{n,m+1} = \phi_{n,m-1}$. With these substitutions, the finite difference equation for point B is

$$2\,\phi_{n-1,m} + 2\,\beta^2\,\phi_{n,m-1} - 2\,(1+\beta^2)\,\phi_{n,m} = 0\,. \qquad [5]$$

* You can verify that at point E (*i = n, j = 1*), the corresponding finite difference equation is

$$2\,\phi_{n-1,1} + 2\,\beta^2\,\phi_{n,2} - 2\,(1+\beta^2)\,\phi_{n,m} = 0\,. \qquad [6]$$

Thus, the numerical solution to the problem has been reduced to the solution of the simultaneous equations given by [1], [2], [3], [4], [5] and [6], with the values of $\phi_{i,j}$ known along lines AF and CD. Using Matlab, we implement the solution by using an explicit numerical scheme. The solution can be represented graphically by using function *contour*. The resulting contour lines represent equipotential lines of the flow.

Calculation of velocities using the velocity potential

Once the values of $\phi_{i,j}$ are known, we can calculate flow velocities using finite-difference approximations for the components, namely, $u = \frac{\partial}{\partial x}\phi$ and $v = \frac{\partial}{\partial y}\phi$. For the interior points of the grid, i.e., for *i = 1, 2, ..., n-1* and *j = 1, 2, ..., m-1*, the velocities are approximated by

$$u_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\,Dx}\ , \text{ and } \ v_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\,Dy}\ .$$

Along the upstream boundary AF (*i = 1*), $v_{i,j} = 0$, and $u_{1,j} = \frac{\phi_{2,j} - \phi_{1,j}}{Dx}$, for *j = 1,2, ..., m* .

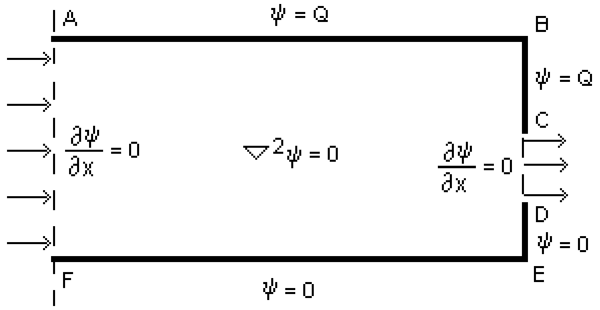
Along the downstream open boundary CD (*i = n*), $v_{i,j} = 0$, and $u_{n,j} = \frac{\phi_{n,j} - \phi_{n-1,j}}{Dx}$, for *j =* $j_1+1\,, j_1+2\,, ..., j_2-1$. Along the top boundary AB (*j=m*), $v_{i,m} = 0$, and $u_{i,m} = \frac{\phi_{i+1,m} - \phi_{i-1,m}}{2\,Dx}$, for *i = 2,3,...,n-1*. Along the bottom boundary FE (*j=1*), $v_{i,1} = 0$, and $u_{i,1} = \frac{\phi_{i+1,1} - \phi_{i-1,1}}{2\,Dx}$, for *i = 2,3,...,n-1*. Along BC and DE (*i=n*), $u_{n,j} = 0$ and $v_{n,j} = \frac{\phi_{n,j+1} - \phi_{n,j-1}}{2\,Dy}$ for *j = 2, 3, ...,* j_1 and *j* = j_2, $j_2 + 1$, ..., $m - 1$. The velocities at points B and E are taken as zero, i.e., $u_{n,1} = v_{n,1} = u_{n,m} = v_{n,m} = 0$. A graph of the velocity field can be produced using Matlab's function *quiver.* Use » *help quiver* for additional information.

Numerical solution for the streamfunction

In this section we consider the solution of the Laplace equation for the streamfunction $\psi(x, y)$. The finite-difference approximation for this equation in the interior points of the solution grid (namely, for *i = 1, 2, ..., n-1* and *j = 1, 2, ..., m-1*) is given by the equations

$$\psi_{i+1,j} + \beta^2\,\psi_{i,j+1} + \phi_{i-1,j} + \beta^2\,\psi_{i,j-1} - 2\,(1 + \beta^2)\,\psi_{i,j} = 0 \qquad [7]$$

where $\beta = \frac{Dx}{Dy}$. With this expression we can generate *(n-2)(m-2)* equations. To complete the solution we need to load the boundary conditions described in the diagram below.



For the Dirichlet-type boundary conditions at boundaries ABC and FED we can write:

* $\psi_{i,1} = 0$ (AB) and $\psi_{i,m} = 0$ (FE) for *i = 1, 2, ..., n*
* $\psi_{n,j} = 0$, for *j = 2,3,...,*j_1 (DE) and for *j* = j_2, $j_2 + 1$,...,*m-1*

For the Neumann-type boundary conditions we need to approximate the derivatives $\frac{\partial}{\partial y}\psi$ with the following finite-differences:

* Along AF, for $\frac{\partial}{\partial x}\psi = 0$, we use $\frac{\psi_{2,j} - \psi_{0,j}}{2\,Dx} = 0$, i.e., $\psi_{0,j} = \psi_{2,j}$. Although the values $\psi_{0,j}$ are off the grid, we can replace them into the finite difference approximation for *i* = 1, namely, $\psi_{2,j} + \beta^2\,\psi_{1,j+1} + \psi_{0,j} + \beta^2\,\psi_{1,j-1} - 2\,(1 + \beta^2)\,\psi_{1,j} = 0$, to produce the set of equations:

$$\beta^2\,\psi_{1,j+1} + 2\,\psi_{2,j} + \beta^2\,\psi_{1,j-1} - 2\,(1 + \beta^2)\,\psi_{1,j} = 0 \ , \qquad [8]$$

for *j = 2, 3, ..., m-1*

* Along CD, for $\frac{\partial}{\partial x}\psi = 0$, we use $\frac{\psi_{n+1,j} - \psi_{n-1,j}}{2\,Dx} = 0$, i.e., $\psi_{n+1,j} = \psi_{n-1,j}$. Although the values $\psi_{n+1,j}$ are off the grid, we can replace them into the finite difference approximation for *i = n*, namely, $\psi_{n+1,j} + \beta^2\,\psi_{n,j+1} + \psi_{n-1,j} + \beta^2\,\psi_{n,j-1} - 2\,(1+\beta^2)\,\psi_{n,j} = 0$, to produce the set of equations:

$$\beta^2\,\psi_{n,j+1} + 2\,\psi_{n-1,j} + \beta^2\,\psi_{n,j-1} - 2\,(1+\beta^2)\,\psi_{n,j} = 0 \ , \qquad [9]$$

for j = $j_1 + 1$, $j_1 + 2$, ..., $j_2 - 1$.

The numerical solution to the problem involves the simultaneous solution of the equations given by [7], [8] and [9], with the values of $\psi_{i,j}$ known along lines ABC and FED. Using Matlab we could implement the solution through an explicit solution scheme for the Laplace equation. The solution can be represented graphically by using function *contour*. The resulting contour lines represent streamlines of the flow.

Calculation of velocities using the streamfunction

Once the values of $\psi_{i,j}$ are known, we can calculate flow velocities using finite-difference approximations for the components, namely, $u = \frac{\partial}{\partial y}\psi$ and $v = -\left(\frac{\partial}{\partial x}\psi\right)$. For the interior points of the grid, i.e., for *i = 1, 2, ..., n-1* and *j = 1, 2, ..., m-1*, the velocities are approximated by

$$u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\,Dy} \ , \text{ and } \ v_{i,j} = \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\,Dx} \ .$$

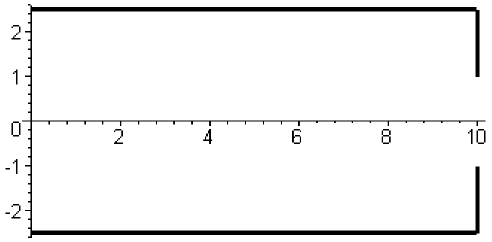
Along the upstream boundary AF (*i = 1*), $v_{i,j} = 0$, and $u_{1,j} = \frac{\psi_{1,j+1} - \psi_{1,j-1}}{2\,Dy}$, for *j = 2,3, ..., m*-1 . Along the downstream open boundary CD (*i = n*), $v_{i,j} = 0$, and $u_{n,j} = \frac{\psi_{n,j+1} - \psi_{n,j-1}}{2\,Dy}$, for *j =* $j_1 + 1$, $j_1 + 2$,...,$j_2 - 1$. Along the top boundary AB (*j=m*), $v_{i,m} = 0$, and $u_{i,m} = \frac{\psi_{i,m} - \psi_{i,m-1}}{Dy}$, for *i = 2,3,...,n-1*. Along the bottom boundary FE (*j=1*), $v_{i,1} = 0$, and $u_{i,1} = \frac{\psi_{i,2} - \psi_{i,1}}{Dx}$, for *i = 2,3,...,n-1*. Along BC and DE (*i=n*), $u_{n,j} = 0$ and $v_{n,j} = \frac{\psi_{n,j} - \psi_{n-1,j}}{Dx}$ for *j = 2, 3, ...,* j_1 and *j =* j_2 , $j_2 + 1$, ..., $m - 1$. The velocities at points B and E are taken as zero, i.e., $u_{n,1} = v_{n,1} = u_{n,m} = v_{n,m}$ = 0. A graph of the velocity field can be produced using Matlab's function *quiver*.

Flow grid display

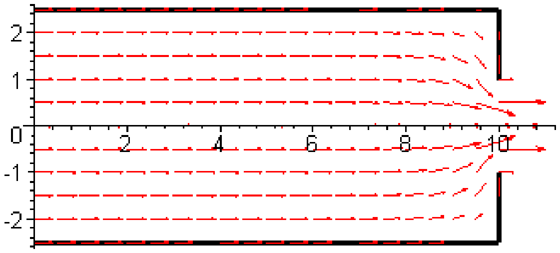
By superimposing the contour lines for velocity potential (equipotential lines) and for streamfunction (streamlines) we can show the flow grid for the problem of interest.

Example solution

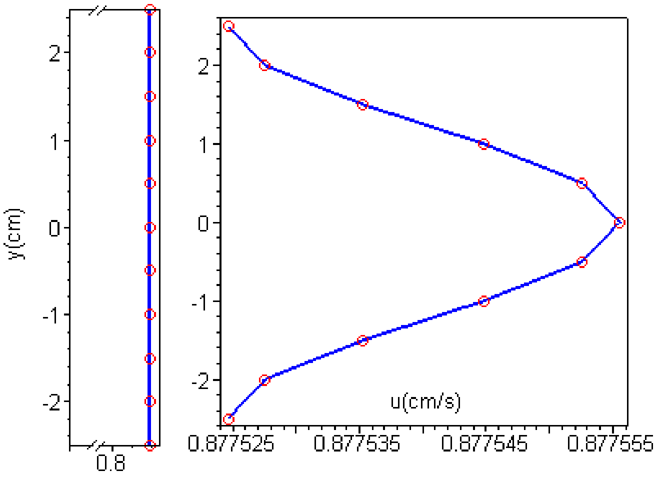
Consider the case in which $w = 10, h = 5, h_0 = 2, Dx = 0.5, Dy = 0.5$. The solution domain would look as illustrated in the following figure.



This solution was achieved using Maple (rather than Matlab), through an implicit solution of the Laplace equation. The velocity vectors calculated by solving for the velocity potential are shown next:

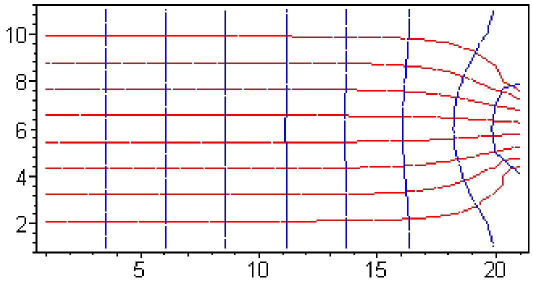


The following diagram shows the velocities at the upstream end calculated from the solution for the velocity potential:



The diagram to the left shows the velocity distribution in a scale of 0 to 9 for the velocity, while, the second one shows a smaller velocity scale. In theory, we should have expected a constant velocity upstream. The small discrepancies in the calculation are due to numerical errors.

The following is the flow grid corresponding to this numerical solution:

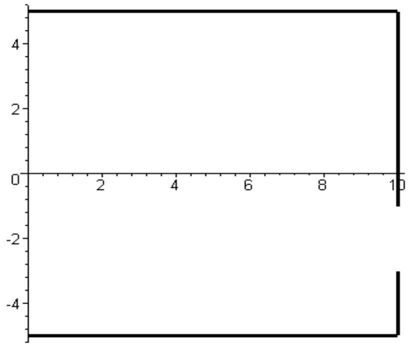


Part 4 – Proposed Exercises

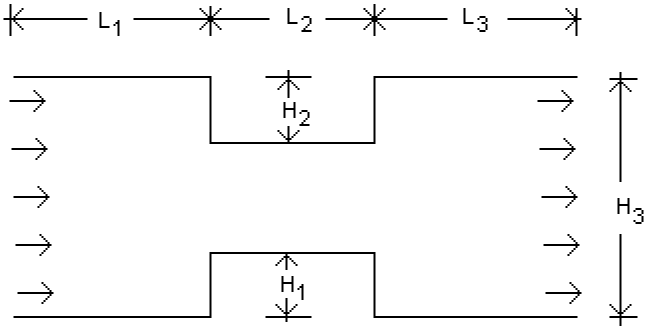
[**1**]. Use an explicit solution of the Laplace equation with the appropriate boundary conditions to solve the problem of potential flow in the domain described above. (**a**) Solve for the velocity potential $\phi(x,y)$, determining velocities, and the discharge at the upstream end if $w = 10$, $h = 5$, $h_0 = 2$, $Dx = 0.5$, $Dy = 0.5$, and the h_o gap is centered in the downstream face. (**b**) Produce drawings showing the equipotential lines and the velocity vectors. (**c**) Using the discharge found in (a), solve for the streamfunction, $\psi(x,y)$, determining velocities for the same solution domain as in (a). (**d**) Produce drawings showing the streamlines and the velocity vectors. (**e**) Produce a graph showing the flow grid (equipotential lines and streamlines).

[**2**]. Repeat problem [1] for $w = 10$, $h = 10$, $h_0 = 2$, $Dx = 0.5$, $Dy = 0.5$. The h_o gap is centered in the downstream face.

[**3**]. Repeat problem [2], but letting the h_o gap be off-center in the downstream face. The top of the gap should be located at a distance of 6 units below the upper right corner of the solution domain, i.e., as illustrated below:

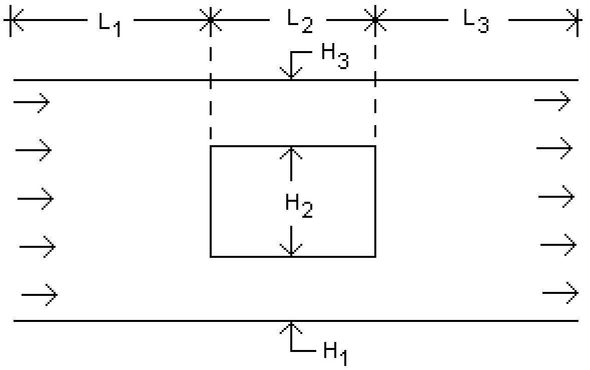


[4]. The figure below shows a two-dimensional potential flow through a rectangular contraction. Let $L_1=10$ in, $L_2=7.5$ in, $L_3 = 10$ in, $H_1 = H_2 = 3$ in, $H_3 = 9$ in. (**a**) Solve for the velocity potential $\phi(x,y)$, determining velocities, and the discharge at the upstream end if $Dx = 0.5$ in, $Dy = 0.5$ in. (**b**) Produce drawings showing the equipotential lines and the velocity vectors. (**c**) Using the discharge found in (a), solve for the streamfunction, $\psi(x,y)$, determining velocities for the same solution domain as in (a). (**d**) Produce drawings showing the streamlines and the velocity vectors. (**e**) Produce a graph showing the flow grid (equipotential lines and streamlines).



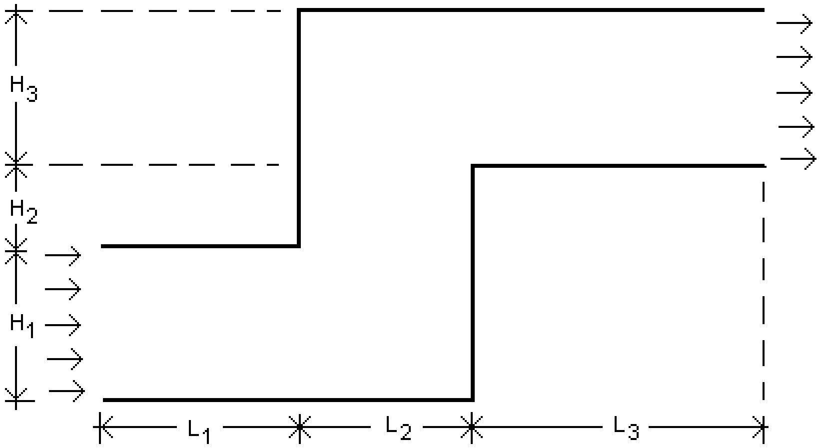
[5]. Repeat problem [4] with the following data: L_1=20 *in*, L_2=5 *in*, L_3 = 10 *in*, H_1 = 4*in*, H_2 = 3 *in*, H_3 = 10 *in*. Use *Dx = 1 in, Dy = 1 in.*

[6]. The figure below shows a two-dimensional potential flow through the gap between two solid walls separated by a distance $H_1+H_2+H_3$, and a rectangular block of width L_2 and height H_2. Let L_1=10 *in*, L_2=7.5 *in*, L_3 = 10 *in*, $H_1 = H_2 = H_3$ = 3 *in*. (**a**) Solve for the velocity potential $\phi(x,y)$, determining velocities, and the discharge at the upstream end if *Dx = 0.5 in, Dy = 0.5 in*. (**b**) Produce drawings showing the equipotential lines and the velocity vectors. (**c**) Using the discharge found in (a), solve for the streamfunction, $\psi(x,y)$, determining velocities for the same solution domain as in (a). (**d**) Produce drawings showing the streamlines and the velocity vectors. (**e**) Produce a graph showing the flow grid (equipotential lines and streamlines).



[7]. Repeat problem [6] with the following data: L_1=20 *in*, L_2=5 *in*, L_3 = 10 *in*, H_1 = 4*in*, $H_2 = H_3$ = 3 *in*. Use *Dx = 1 in, Dy = 1 in.*

[8]. The figure below shows a two-dimensional potential flow through a double rectangular elbow. Let L_1=10 *in*, L_2=7.5 *in*, L_3 = 10 *in*, $H_1 = H_2 = H_3$ = 3 *in*. (**a**) Solve for the velocity potential $\phi(x,y)$, determining velocities, and the discharge at the upstream end if *Dx = 0.5 in, Dy = 0.5 in*. (**b**) Produce drawings showing the equipotential lines and the velocity vectors. (**c**) Using the discharge found in (a), solve for the streamfunction, $\psi(x,y)$, determining velocities for the same solution domain as in (a). (**d**) Produce drawings showing the streamlines and the velocity vectors. (**e**) Produce a graph showing the flow grid (equipotential lines and streamlines).



[9]. Repeat problem [6] with the following data: $L_1=20$ *in*, $L_2=5$ *in*, $L_3 = 10$ *in*, $H_1 = 4$*in*, $H_2 = H_3 = 3$ *in*. Use $Dx = 1$ *in*, $Dy = 1$ *in*.