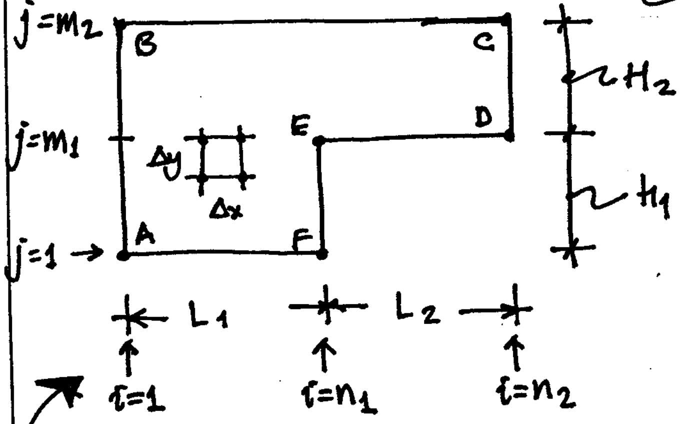
No. 937 811E Engineer's Computation Pad
STAEDTLER

Problem in Vreugdenhill's book, p. 146 – DETAILS OF SOLUTION

USE $h = 4\,m$
$U = 5\,m/s$

Domain dimensions: $L_1 = L_2 = h$, $H_1 = H_2 = h/2$



CASES TO TRY:

Case 1 → $\Delta x = \Delta y = h/4 = 1\,m$

Case 2 → $\Delta x = \Delta y = h/8 = 0.5\,m$

Case 3 → $\Delta x = \Delta y = h/16 = 0.25\,m$

INDICES FOR GRID: $n_1 = fix\left(\frac{L_1}{\Delta x}\right) + 1$ $\quad n_2 = fix\left(\frac{L_2}{\Delta x}\right) + n_1$

$m_1 = fix\left(\frac{H_1}{\Delta y}\right) + 1$ $\quad m_2 = fix\left(\frac{H_2}{\Delta y}\right) + m_1$

GOVERNING EQUATION: $\nabla^2 \psi = 0$ or $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

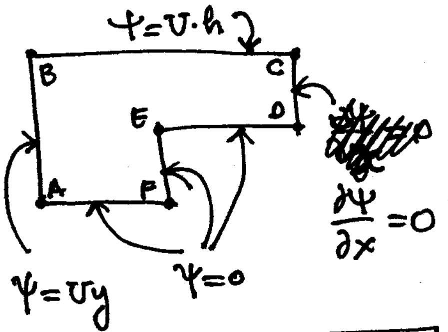
FINITE-DIFFERENCE APPROXIMATION FOR EXPLICIT SOLUTION:

$$\psi_{ij}^{N} = \frac{\psi_{i-1,j} + \psi_{i+1,j} + \beta^2(\psi_{i,j-1} + \psi_{i,j+1})}{2(1+\beta^2)}, \quad \beta = \frac{\Delta x}{\Delta y}$$

The "N" super-index indicates new value in iterative process

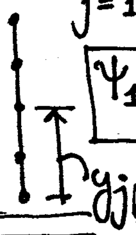
NOTE:
In programming iterative process refer to ψ_{ij} as Psi(i,j) and to ψ_{ij}^{N} as PsiN(i,j)

Boundary conditions → In programming



FOR $i = 2, 3, \ldots, n_1 - 1$
$j = 2, 3, \ldots, m_2 - 1$,
and FOR
$i = n_1, n_1+1, \ldots, n_2 - 1$
$j = m_1+1, m_1+2, \ldots, m_2 - 1$

AB FOR $i = 1$,
$j = 1, 2, \ldots, m_2$



$\boxed{\psi_{1,j} = U \cdot y_j}$

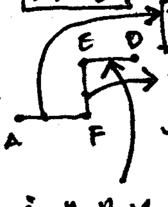
BC

FOR $i = 1, 2, \ldots, n_2$
$j = m_2$

$\boxed{\psi_{i,m_2} = Uh}$

AFED $j = 1$
$i = 1, 2, \ldots, n_1$

$\boxed{\psi_{i,1} = 0}$



$i = n_1$
$j = 1, 2, \ldots, m_1$

$\boxed{\psi_{n_1,j} = 0}$

$i = n_1, n_1+1, \ldots, n_2$, $j = m_1$

$\boxed{\psi_{i,m_1} = 0}$

CD $i = n_2$, $j = m_1, m_1+1, \ldots, m_2$

$$\frac{\partial \psi}{\partial x} = \frac{\psi_{n_2,j} - \psi_{n_2-1,j}}{\Delta x} = 0$$

$$\Rightarrow \boxed{\psi_{n_2,j} = \psi_{n_2-1,j}}$$

ALGORITHM TO SOLVE FOR ψ_{ij}

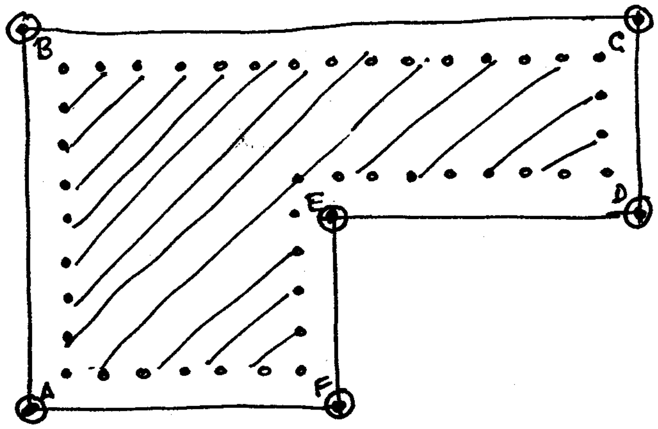
1. DATA: $L_1, L_2, H_1, H_2, \Delta x, \Delta y, U, h$
2. CALCULATE n_1, n_2, m_1, m_2
3. CREATE MATRICES ψ_{ij} and ψ_{ij}^N, all equal to zero for
   $\left\{\begin{array}{l} i = 1,2,\dots,n_1 \\ j = 1,2,\dots,m_2 \end{array}\right\}$ and $\left\{\begin{array}{l} i = n_1+1,\dots,n_2 \\ j = m_1, m_1+1,\dots,m_2 \end{array}\right\}$

   NOTE: In Matlab, use NAN (Not-a-number) to fill nodes not involved in solution
4. LOAD BOUNDARY CONDITIONS FOR NEW VALUE OF ψ, i.e., for ψ_{ij}^N along boundaries
5. CALCULATE ψ_{ij}^N USING FINITE-DIFFERENCE SCHEME
6. For all values of i,j relevant to problem calculate $e_{ij} = |\psi_{ij}^N - \psi_{ij}|$
7. If all e_{ij}'s are such that $e_{ij} < \varepsilon$, ε = Tolerance for convergence $\Rightarrow$ solution has been found - end process; else ~~repeat steps~~ (use $\varepsilon = 0.001$, for example)
8. For all values of ij relevant to solution, make $\psi_{ij} = \psi_{ij}^N$, and repeat steps 4 through 8 until convergence, or until a maximum allowed number of iterations is reached (imax = 1000, for example)

ITERATIVE PROCESS (STEPS 4-8)

VELOCITY CALCULATIONS

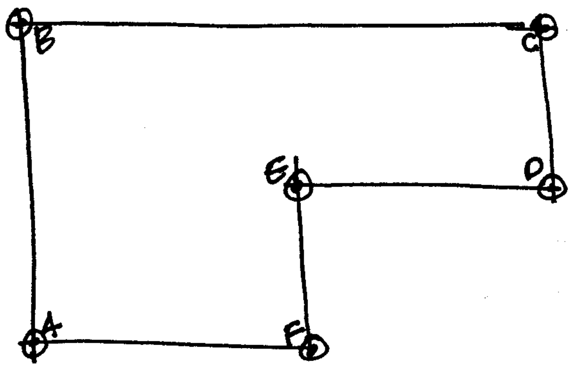
$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$



INTERIOR POINTS (shaded area)

$$u = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}$$

$$v = -\left(\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}\right)$$



ALONG BOUNDARIES (but not including corner points)

Along AB

$$u = \frac{\psi_{1,j+1} - \psi_{1,j-1}}{2\Delta y}, \quad j = 2,3,\ldots,m_2-1$$

$$v = -\left(\frac{\psi_{2,j} - \psi_{1,j}}{\Delta x}\right)$$

CORNER (A)

$$u = \frac{\psi_{1,2} - \psi_{1,1}}{\Delta y}$$

$$v = -\left(\frac{\psi_{2,1} - \psi_{1,1}}{\Delta x}\right)$$

CORNER (B)

$$u = \frac{\psi_{1,m_2} - \psi_{1,m_2-1}}{\Delta y}$$

$$v = -\left(\frac{\psi_{2,m_2} - \psi_{1,m_2}}{\Delta x}\right)$$

ALONG BC $i = 2,3,\ldots,n_2-1$

$$u = \frac{\psi_{i,m_2} - \psi_{i,m_2-1}}{\Delta y}$$

$$v = -\left(\frac{\psi_{i+1,m_2} - \psi_{i-1,m_2}}{2\Delta x}\right)$$

CORNER (C)

$$u = \frac{\psi_{n_2,m_2} - \psi_{n_2,m_2-1}}{\Delta y}$$

$$v = -\left(\frac{\psi_{n_2,m_2} - \psi_{n_2-1,m_2}}{\Delta x}\right)$$

CORNER (D)

$$u = \frac{\psi_{n_2,m_1+1} - \psi_{n_2,m_1}}{\Delta y}$$

$$v = -\left(\frac{\psi_{n_2,m_1} - \psi_{n_2-1,m_1}}{\Delta x}\right)$$

ALONG ED $i = n_1+1,\ldots,n_2-1$

$$u = \frac{\psi_{i,m_1+1} - \psi_{i,m_1}}{\Delta y}$$

$$v = \frac{\psi_{i+1,m_1} - \psi_{i-1,m_1}}{2\Delta x}$$

CORNER (E)

$$u = \frac{\psi_{n_1,m_1+1} - \psi_{n_1,m_1-1}}{2\Delta y}$$

$$v = -\left(\frac{\psi_{n_1+1,m_1} - \psi_{n_1-1,m_1}}{2\Delta x}\right)$$

CORNER (F)

$$u = \frac{\psi_{n_1,2} - \psi_{n_1,1}}{\Delta y}$$

$$v = \frac{\psi_{n_1,1} - \psi_{n_1-1,1}}{\Delta x}$$

ALONG CD $j = m_1+1,\ldots,m_2-1$

$$u = \frac{\psi_{n_2,j+1} - \psi_{n_2,j-1}}{2\Delta y}$$

$$v = -\left(\frac{\psi_{n_2,j} - \psi_{n_2-1,j}}{\Delta x}\right)$$

ALONG EF, $j = 2,\ldots,m_1-1$

$$u = \frac{\psi_{n_1,j+1} - \psi_{n_1,j-1}}{2\Delta y}$$

$$v = -\left(\frac{\psi_{n_1,j} - \psi_{n_1-1,j}}{\Delta x}\right)$$

ALONG AF

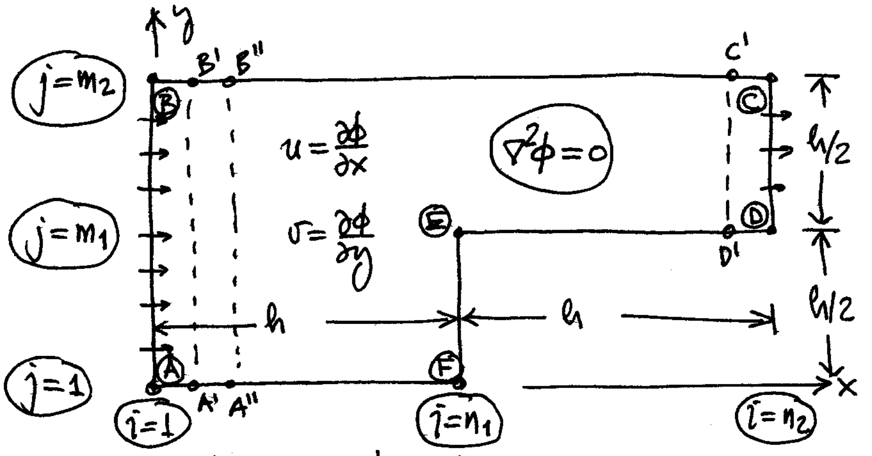
$$u = \frac{\psi_{i,2} - \psi_{i,1}}{\Delta y}$$

$$v = -\left(\frac{\psi_{i+1,1} - \psi_{i-1,1}}{2\Delta x}\right)$$

- USE function "quiver" to plot velocity vectors
- USE function "contour" to plot streamlines.

Problem in Yreughill's book, p. 146 — SOLUTION USING VELOCITY POTENTIAL (ϕ)

We use same solution domain as used to solve in terms of ψ. In terms of ϕ, the boundary conditions are as follows:



ALONG SOLID BOUNDARIES (BC, AF, FE, ED), normal velocity is zero, i.e., no flow across those boundaries, thus:

$\boxed{BC}$ $v = \dfrac{\partial\phi}{\partial y} = 0$, or $\dfrac{\phi_{i,m_2} - \phi_{i,m_2-1}}{\Delta y} = 0$

$\Rightarrow \boxed{\phi_{i,m_2} = \phi_{i,m_2-1}}$ FOR $i = 2, 3, \ldots, n_2-1$

$\boxed{AF}$ $v = \dfrac{\partial\phi}{\partial y} = 0$, or $\dfrac{\phi_{i,2} - \phi_{i,1}}{\Delta y} = 0 \Rightarrow \boxed{\phi_{i,1} = \phi_{i,2}}$ FOR $i = 2, 3, \ldots, n_1$

$\boxed{FE}$ $u = \dfrac{\partial\phi}{\partial x} = 0$, or $\dfrac{\phi_{n_1,j} - \phi_{n_1-1,j}}{\Delta x} = 0 \Rightarrow \boxed{\phi_{n_1,j} = \phi_{n_1-1,j}}$ FOR $j = 2, 3, \ldots, m_1-1$

$\boxed{ED}$ $v = \dfrac{\partial\phi}{\partial y} = 0$, or $\dfrac{\phi_{i,m_1+1} - \phi_{i,m_1}}{\Delta y} = 0 \Rightarrow \boxed{\phi_{i,m_1} = \phi_{i,m_1+1}}$ FOR $i = n_1+1, \ldots, n_2-1$

<u>On upstream section (inlet) AB</u>: $u = \dfrac{\partial\phi}{\partial x} = U$ (constant velocity) and $v = \dfrac{\partial\phi}{\partial y} = 0$

Since $\partial\phi/\partial y = 0$, then ϕ is a function of x only. Integrating $\dfrac{\partial\phi}{\partial x} = U \Rightarrow \phi_{AB} = Ux + C$

Now, at section AB, we select $x = 0$, thus $\boxed{\phi_{AB} = C}$

<u>On downstream section (outlet) CD</u>: $u = \dfrac{\partial\phi}{\partial x} = U'$ (unknown constant velocity) and $\dfrac{\partial\phi}{\partial y} = 0$.

Thus, $\phi_{CD} = U'x + C$. Since at CD, $x = 2h$, $\phi_{CD} = 2hU' + C$.

By continuity, $Q = U \cdot h = U' \cdot h/2 \Rightarrow U' = 2U$, and $\boxed{\phi_{CD} = 4hU + C}$

<u>Sign of ϕ</u>: at section AB, $u = U > 0 \Rightarrow \dfrac{\partial\phi}{\partial x} > 0$ or $\phi_{AB} < \phi_{A'B'} < \phi_{A''B''}$. Also, at section CD, $u = U' > 0 \Rightarrow \phi_{C'D'} < \phi_{CD}$. In summary, ϕ must increase as we move from AB to CD, and $\phi_{AB} < \phi_{CD}$. Let's take $\underline{\phi_{CD} = 4hU + C = 0}$, so that $C = -4hU$, and, therefore, $\underline{\phi_{AB} = -4hU}$

Thus: AB $\rightarrow \boxed{\phi_{1,j} = -4hU}$ FOR $j = 1, 2, \ldots, m_2$ and CD $\rightarrow \boxed{\phi_{n_2,j} = 0}$, FOR $j = m_1, m_1+2, \ldots, m_2$

Calculation of ϕ_{ij} for <u>interior points</u>, i.e. for the blocks:

$$\left\{\begin{array}{c} i = 2,3,\dots,n_1-1 \\ \text{and} \\ j = 2,3,\dots,m_2-1 \end{array}\right\} \text{ and } \left\{\begin{array}{c} i = n_1, n_1+1,\dots,n_2-1 \\ \text{and} \\ j = m_1+1, m_1+2,\dots,m_2 \end{array}\right\} \text{ use:}$$

$$\boxed{\phi_{ij}^{N} = \frac{\phi_{i-1,j} + \phi_{i+1,j} + \beta^2(\phi_{i,j-1} + \phi_{i,j+1})}{2(1+\beta^2)}}, \quad \beta = \frac{\Delta x}{\Delta y}$$

Thus, in programming, use matrices Phi (i,j) for ϕ_{ij} and PhiN(i,j) for ϕ_{ij}^{N}.

<u>ALGORITHM</u>: Same as when solving for ψ, but with different boundary conditions.

<u>VELOCITIES</u>: To calculate velocities use finite differences as when solving for ψ, but recalling that, while using ϕ, $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$.

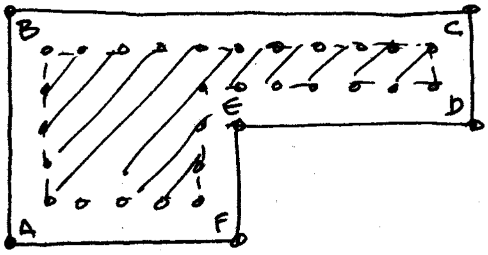
To simplify obtaining the corresponding equations for u_{ij}, v_{ij}, take the equations for u_{ij}, v_{ij} in terms of ψ and make the following changes:

- replace ψ with ϕ
- change sign of the equations for v
- change u for v and vice versa

Thus, FOR INTERIOR POINTS (shaded area) $\Longrightarrow$

$$v = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y}$$

$$u = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}$$



For example, along CD, $j = m_1+1,\dots,m_2-1$

$$v = \frac{\phi_{n_2,j+1} - \phi_{n_2,j-1}}{2\Delta y}$$

$$u = \frac{\phi_{n_2,j} - \phi_{n_2-1,j}}{\Delta x}$$