

# Numerical solution to unsteady two-dimensional Poiseuille flow

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## Description of problem

This is a problem in which the resulting governing equation is a type of diffusion equation. The general equations that govern unsteady two-dimensional flow are the Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (2)$$

and the continuity equation for incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

In equations (1) through (3),  $u$  and  $v$  are the velocity components of the flow in the  $x$  and  $y$  directions, respectively,  $p$  is the pressure,  $t$  is time,  $\rho$  is density (e.g., for water at  $20^\circ\text{C}$ ,  $\rho = 1000 \text{ kg/m}^3$ ),  $\nu$  is the kinematic viscosity (e.g., for water at  $20^\circ\text{C}$ ,  $\nu = 1 \times 10^{-6} \text{ m}^2/\text{s}$ ),

The Poiseuille flow is a laminar flow between two parallel walls driven by a constant pressure gradient  $-\partial p/\partial x = K$ . The flow is essentially horizontal, which means  $v = 0$  and  $u = u(y, t)$ . The condition  $v = 0$ , replaced into (3), produces  $\partial u/\partial x = 0$ . These two conditions (namely,  $v = 0$  and  $\partial u/\partial x = 0$ ), allows us to simplify equations (1) and (2) to the following:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (4)$$

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (5)$$

Equation (5) indicates that  $p$  is not a function of  $y$ , thus,  $p = p(x)$ , and

$$\partial p/\partial x = dp/dx = -K, \quad (6)$$

Thus, the resulting governing equation is obtained by combining (6) and (4):

$$\frac{\partial u}{\partial t} = \frac{K}{\rho} + \nu \frac{\partial^2 u}{\partial y^2}. \quad (7)$$

Equation (7) can be used to describe an unsteady Poiseuille flow such that the fluid is initially at rest, i.e.,  $u(y,0) = 0$ , and it's suddenly subject to the pressure gradient  $K$ . Let the coordinate system be centered in the middle of the two-dimensional channel of width  $2b$ , such that the walls of the channel are located at  $y = +b$  and  $y = -b$ . The so-called *no-slip* boundary condition requires that the velocity of the fluid particles in contact with a solid wall have the same velocity of the wall. Since the walls for a Poiseuille flow are at rest, boundary conditions for this problem would be  $u(-b,t) = 0$  and  $u(+b,t) = 0$ .

### Steady-state solution

The unsteady problem described above is also referred to as the problem of flow establishment, i.e., the description of the flow velocity as the steady pressure gradient is applied and before a steady state is reached. A steady state will be reached as  $t \rightarrow \infty$ , i.e., when  $\partial u / \partial t = 0$ , a situation for which the governing equation becomes

$$0 = \frac{K}{\rho} + \nu \frac{\partial^2 u}{\partial y^2}. \quad (8)$$

For the steady state conditions,  $u$  is not longer a function of  $t$ , i.e.,  $u = u(y)$ , and the partial derivative in (8) can be converted into a total derivative, i.e.,

$$\frac{d^2 u}{dy^2} = -\frac{K}{\mu}, \quad (9)$$

where  $\mu = \rho \cdot \nu$  is the dynamic (or absolute) viscosity of the fluid. Since the right-hand side of equation (9) is constant, the integration of this equation is straightforward:

$$\frac{du}{dy} = -\frac{K}{\mu} y + C_1, \quad (10)$$

and

$$u(y) = -\frac{K}{2\mu} y^2 + C_1 \cdot y + C_2, \quad (11)$$

where  $C_1$  and  $C_2$  are integration constants. Replacing the boundary conditions

$$u(b) = u(-b) = 0$$

into equation (11), produce the equations:

$$0 = -\frac{Kb^2}{2\mu} + C_1 \cdot b + C_2, \quad (12)$$

$$0 = -\frac{Kb^2}{2\mu} - C_1 \cdot b + C_2, \quad (13)$$

Solving for the constants of integration produces:

$$C_1 = 0, \quad C_2 = \frac{Kb^2}{2\mu} \quad (14)$$

Replacing the results in (14) into the equation for the steady velocity distribution produces the parabolic function:

$$u(y) = \frac{Kb^2}{2\mu} - \frac{K}{2\mu} y^2 = \frac{Kb^2}{2\mu} \left( 1 - \left( \frac{y}{b} \right)^2 \right), \quad (11)$$

A dimensionless solution can be produced by introducing the following dimensionless variables:

$$U = \frac{\mu \cdot u(y)}{Kb^2}, \quad Y = \frac{y}{b}, \quad (12)$$

into (11). The result is:

$$U = \frac{1}{2}(1 - Y^2), \quad -1 \leq Y \leq 1 \quad (13).$$

### Dimensionless unsteady problem

To solve the unsteady problem described by equation (7), we use the dimensionless variables described in (12), plus a dimensionless time:

$$T = \frac{\nu \cdot t}{b^2}. \quad (14)$$

With these dimensionless variables, equation (7) becomes

$$\frac{\partial U}{\partial T} = 1 + \frac{\partial^2 U}{\partial Y^2}, \quad -1 \leq Y \leq 1, \quad (7)$$

where  $U = U(Y, T)$ . The dimensionless initial conditions are  $U(Y, 0) = 0$ , while the dimensionless boundary conditions are  $U(1, T) = U(-1, T) = 0$ .

### Numerical solution

The numerical solution will be carried out in the spatial domain  $-1 \leq Y \leq 1$ , and in the time domain  $0 \leq T \leq T_{\max}$ . Theoretically,  $T_{\max}$  should be infinity, however, for all

practical applications a dimensionless time of  $T_{max} = 3$  is sufficient to show the unsteady behavior of the dimensionless velocity  $U$ .

The numerical scheme proposed to use for this solution is Crank-Nicholson scheme in which the time derivative is replaced by a forward difference, namely,

$$\frac{\partial U}{\partial T} \approx \frac{U_j^{n+1} - U_j^n}{\Delta T}, \quad (8)$$

and the second-order time derivative is replaced by the formula

$$\frac{\partial^2 U}{\partial T^2} \approx \theta \cdot \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta X^2} + (1-\theta) \cdot \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta X^2}. \quad (9)$$

The resulting finite-difference scheme is, therefore,

$$\frac{U_j^{n+1} - U_j^n}{\Delta T} = 1 + \theta \cdot \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta X^2} + (1-\theta) \cdot \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta X^2}. \quad (10)$$

### Exercise: Implicit scheme

(a) Write out the equation for the dimensionless velocities  $U_j^{n+1} = U(Y_j, T_{n+1})$  in the form of a tri-diagonal system:

$$a_j U_{j-1}^{n+1} + b_j U_j^{n+1} + c_j U_{j+1}^{n+1} = d_j. \quad (11)$$

Obtain expressions for  $a_j$ ,  $b_j$ ,  $c_j$ , and  $d_j$  so that you can fill out the resulting tri-diagonal matrix in a manner resembling that of equation (7.14), page 40, in Vreugdenhill's *Computational Hydraulics*.

(b) Modify function *thomas.m* (available in the file *CompHydExVI\_Script.zip* that you can download from the *Assignment 7* web page) to solve the problem of developing Poiseuille flow described above. Notice that the initial conditions are the same, but there is only one type of boundary conditions (Dirichlet boundary conditions) at positions  $Y=-1$  and  $Y=+1$ , i.e.,  $U = 0$ . Use  $T_{max} = 3.0$ , and select values of  $\Delta X$  and  $\Delta T$  so that the stability condition  $0 < \lambda < 1$  is fulfilled. Here,  $\lambda = \Delta T / \Delta X^2$ .

(c) Produce a movie of the velocity distribution as function of time.