Background material for the Matlab script "CompHyd0_Script.m" based on Chapter 2 from Vreugdenhil, C.B., 1989, "Computational Hydraulics - An Introduction," Springer-Verlag, Berlin.

Material from Chapter 2 - Water Quality in a Lake

Consider a lake with a single input discharge, Q_i, carrying a contaminant (say, BOD) at a concentration c_i. The average concentration of contaminant in the lake is c. The lake has an output discharge of Q_o at a concentration c, the same as the lake concentration. We can also include in the balance an amount L representing the amount of contaminant discharged directly in the lake (say, by dumping contaminant). The volume of the lake is V, and the rate of degradation is given by Vc/T_r, where T_r is a degradation time scale. The mass of contaminant in the lake at any given time is given by Vc. Thus, the rate of change of that mass is given by the equation

$$\frac{d}{dt}(Vc) = Q_i c_i + L - Q_o c - \frac{Vc}{T_r}.$$

For equal inflow and outflow volumes $Q_i = Q_o$, the equation simplifies to

$$\frac{d}{dt}(Vc) = Q(c_i - c) + L - \frac{Vc}{T_r}.$$

What we need to know to solve this equation:

- The initial concentration, c_o
- The river discharge, Q
- The inflow concentration, c_i
- The magnitude of the of the direct discharge, L

If the system reaches an equilibrium condition, then $dc/dt = 0$, and the equation produces the equilibrium concentration

$$c_e = \frac{L + Q_i c_i}{Q_o + \frac{V}{T_r}}.$$

With this value, we can write the governing equation as

$$\frac{dc}{dt} + \frac{c}{T} = \frac{c_e}{T},$$

where

$$T = \frac{V}{Q_i + \frac{V}{T_r}}.$$

Notice that if $Q_i = 0$, $T = T_r$, i.e., the degradation time scale. On the other hand, if $T_r \rightarrow \infty$, $T = V/Q_i$, the flushing time.

With $c(0) = c_o$, an analytical solution to the governing equation is

$$c(t) = c_e + (c_o - c_e)e^{-t/T}.$$

This equation can be found with Matlab by using function *dsolve*. Data used to plot the exact solution is given in the textbook and used in the script *CompHydEx0_Script.m*.

The textbook illustrates the solution for an oscillatory external excitation (i.e., *ce(t)*) in figures 2.3 and 2.4, page 8. These figures are reproduced in the script by taking the external excitation as

$$ce(t) = ce_0 + ce_1 \cdot \cos\left(\frac{2\pi t}{T_e}\right),$$

where ce_0 and ce_1 are constant values, and T_e is the time scale (period) of the oscillatory external excitation. The solution, as presented by Matlab -- using function *pretty* -- is shown as:

```
» pretty(sol)

  /      2           2  2         2       pi t                        pi t
  |ce0 Te  + 4 ce0 pi  T  + ce1 Te  cos(2 ----) + 2 T ce1 pi Te sin(2 ----) +
  \                                         Te                          Te

                           2           2  2         2        2          2  2
        exp(- t/T) (-ce0 Te  - 4 ce0 pi  T  - ce1 Te  + c0 Te  + 4 c0 pi  T )

          2   /    2       2  2
        Te   /  (Te  + 4 pi  T ) + 4 exp(- t/T)
            /

                2           2  2         2        2          2  2    2  2   /
        (-ce0 Te  - 4 ce0 pi  T  - ce1 Te  + c0 Te  + 4 c0 pi  T ) pi  T   /
                                                                          /

           2       2  2 \   /    2       2  2
        (Te  + 4 pi  T )|  /  (Te  + 4 pi  T )
                        / /
```

This expression can be written in a simpler manner by defining the following terms:

$$T_s^2 = T_e^2 + 4\pi^2 T^2,$$

$$f_s(t) = ce_0 \cdot T_s^2 + ce_1 \cdot T_e^2 \cdot \cos\left(\frac{2\pi t}{T_e}\right) + 2\pi \cdot ce_1 \cdot T_e \cdot T \cdot \sin\left(\frac{2\pi t}{T_e}\right),$$

$$f_1 = T_s^2 \cdot (c_0 - ce_0) - ce_1 \cdot T_e^2,$$

so that

$$c(t) = \frac{f_s(t) + f_1 \cdot \exp(-t/T)}{T_s^2}.$$

Notice that the solution consists of a decay term, namely, $f_1\ exp(-t/T)\ /T_s^2$, and a steady-state term, namely, $f_s(t)\ /T_s^2$. The steady state term is a sinusoidal signal, as shown in Figure 2.3, page 8, in Vreugdenhil's book. For the script, the values used to reproduce Figure 2.3 are: $c_0 = 0.2c_e$, where c_e is the equilibrium concentration for the constant-inflow concentration case, $ce_0 = c_e$, $ce_1 = ce/5$, and $T_e = 30\ days$.

The script reproduces Figures 2.3 and 2.4 for the book. Notice, however, that the c axes in those figures in the book actually represent a fraction of concentration (most likely, c/c_0), while those produced by the script show the actual concentration values in *mg/lt*.

Figure 2.4 illustrates the fact that, for an oscillatory external excitation, the initial concentration c_0 has not a significant effect on the solution by showing solutions corresponding to $c_e = 0$ and $c_0 = 0.5c_e$. For both initial concentrations, the signal quickly reach the same oscillatory behavior.

This document was prepared by Gilberto E. Urroz, Associate Professor, Department of Civil and Environmental Engineering, Utah State University, on September 14, 2004. The solutions are based on the book Vreugdenhil, C.B., 1989, "Computational Hydraulics - An Introduction," Springer-Verlag, Berlin.