Notes on the use of double integrals for calculating area properties and volumes

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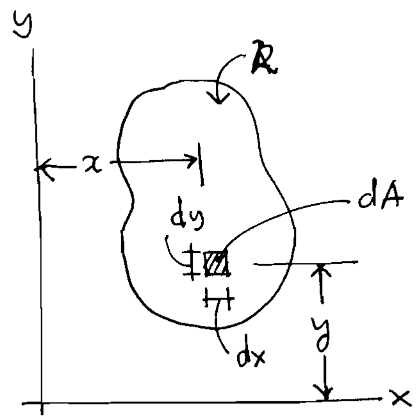
for

CEE 3500 – CEE Fluid Mechanics

Double Integrals – Use them for calculating area properties, for example. The figure shows a Cartesian element of area, $dA = dx \cdot dy$.



The area of the region R is calculated as

$$A = \iint_R dA = \iint_R dy\,dx$$

The first moments of area are

$$M_x^A = \iint_R y\,dA = \iint_R y\,dy\,dx$$

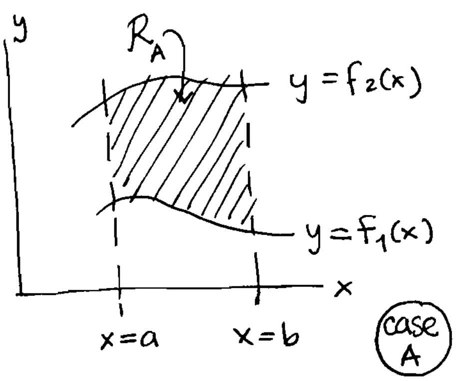
$$M_y^A = \iint_R x\,dA = \iint_R x\,dy\,dx$$

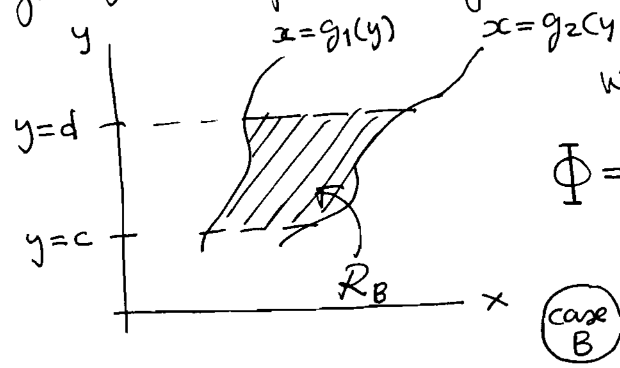
and the moments of inertia are $I_x = \iint_R y^2 dA = \iint_R y^2 dy\,dx$,

$$I_y = \iint_R x^2 dA = \iint_R x^2 dy\,dx, \quad I_o = \iint_{R^2} r^2 dA = \iint_{R^2} (x^2+y^2)\,dy\,dx$$

Actual calculation of double integrals is performed by using iterated integrals:

First, we need to define the region R in two possible ways ↴





we'll calculate

$$\Phi = \iint_R \phi(x,y)\,dA$$

GENERIC DOUBLE INTEGRAL

$$R_A = \{ f_1(x) \le y \le f_2(x),\ a \le x \le b \}$$

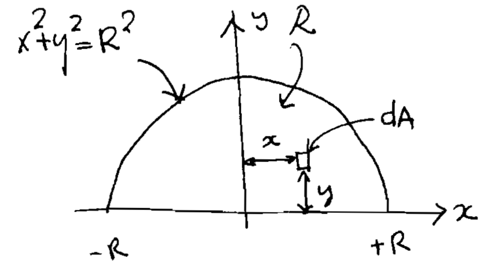
$$R_B = \{ g_1(y) \le x \le g_2(y),\ c \le y \le d \}$$

$$\text{case A:}\quad \Phi = \iint_{R_A} \phi(x,y)\,dA = \int_a^b \underbrace{\int_{f_1(x)}^{f_2(x)} \phi(x,y)\,dy}_{\substack{\text{calculate first} \\ = \psi(x)}}\,dx = \int_a^b \psi(x)\,dx$$

NOTE THE ORDER OF dx, dy

$$\text{case B:}\quad \Phi = \iint_{R_B} \phi(x,y)\,dA = \int_c^d \underbrace{\int_{g_1(y)}^{g_2(y)} \phi(x,y)\,dx}_{\substack{\text{calculate first} \\ = \eta(y)}}\,dy = \int_c^d \eta(y)\,dy$$

EXAMPLE $\quad R=\{0\le y\le\sqrt{R^2-x^2},\ -R\le x\le R\}$, $dA = dy\,dx$



$$A=\iint_R dA = \int_{-R}^{+R}\int_0^{\sqrt{R^2-x^2}} dy\,dx = \frac{\pi R^2}{2}$$

USING HP-49G+: $\displaystyle\int_{-R}^{R}\int_0^{\sqrt{R^2-x^2}} 1\,dy\,dx = \frac{R^2\cdot\pi}{2}$

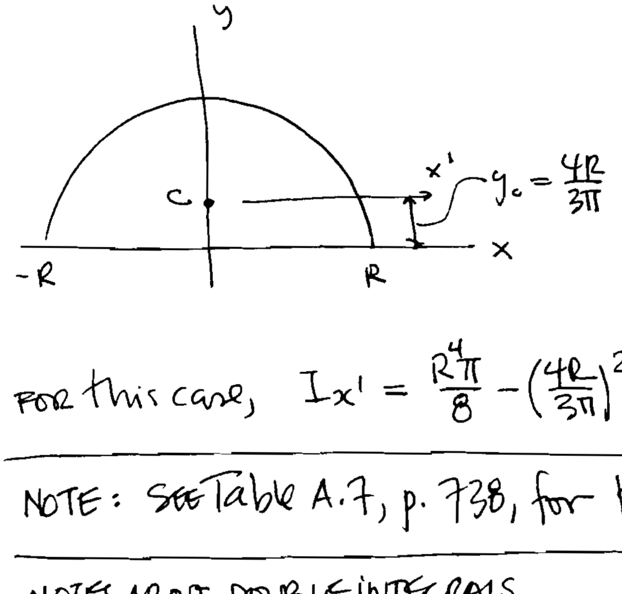
$$M_x^A = \iint_R y\,dA = \int_{-R}^{R}\int_0^{\sqrt{R^2-x^2}} y\,dy\,dx = \frac{2R^3}{3},\quad M_y^A = \iint_R x\,dA = \int_{-R}^{R}\int_0^{\sqrt{R^2-x^2}} x\,dy\,dx = 0$$

$$I_x^A = \iint_R y^2\,dA = \int_{-R}^{R}\int_0^{\sqrt{R^2-x^2}} y^2\,dy\,dx = \frac{R^4\pi}{8},\quad I_y^A = \iint_R x^2\,dA = \int_{-R}^{R}\int_0^{\sqrt{R^2-x^2}} x^2\,dy\,dx = \frac{R^4\pi}{8}$$

CENTROID: $\displaystyle x_c = \frac{M_y^A}{A} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{0}{\pi R^2/2} = 0,\quad y_c = \frac{M_x^A}{A} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{2R^3/3}{\pi R^2/2} = \frac{4R}{3\pi}$

radii of gyration: $\displaystyle k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{R^4\pi/8}{R^2\pi/2}} = \frac{R}{2},\quad k_y = \frac{R}{2}$

polar moment of inertia and radius of gyration: $\displaystyle I_0 = I_x + I_y = \frac{R^4\pi}{4},\quad k_0 = \sqrt{\frac{I_0}{A}} = \frac{\sqrt{2}}{2}R$



let x' = centroidal x axis, i.e., x-axis that passes through centroid. The parallel axes theorem indicates that

$$I_x = I_{x'} + y_c^2 A, \text{ i.e.,}$$

$$I_{x'} = I_x - y_c^2 A$$

FOR this case, $\displaystyle I_{x'} = \frac{R^4\pi}{8} - \left(\frac{4R}{3\pi}\right)^2\cdot\frac{R^2\pi}{2} = R^4\left(\frac{9\pi^2-64}{72}\right)$

NOTE: SEE Table A.7, p. 738, for properties of areas

NOTES ABOUT DOUBLE INTEGRALS

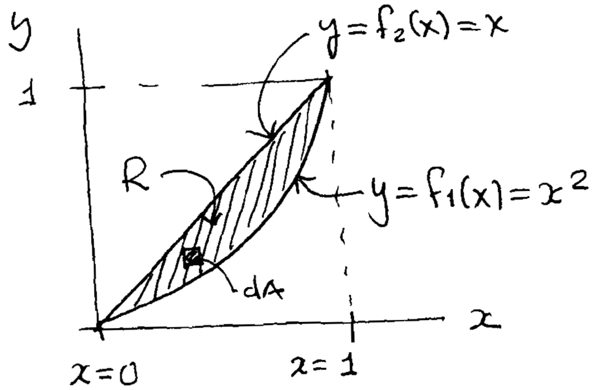
① If R is rectangular, $R=\{c\le y\le d,\ a\le x\le b\}$, then $\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$, i.e., order of integration can be exchanged

② If R is rectangular (see above) and $f(x,y) = h(x)g(y)$, then $\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\int_c^d h(x)\cdot g(y)\,dy\,dx = \left[\int_a^b h(x)\,dx\right]\left[\int_c^d g(y)\,dy\right]$

NOTE: $\phi(x,y)$ is any function of the independent variables x, y. When calculating area properties we can take:

- FOR AREA, $\phi(x,y) = 1.0$
- FIRST MOMENTS, $\phi(x,y) = x$ or $\phi(x,y) = y$
- MOMENTS of INERTIA, $\phi(x,y) = x^2$, $\phi(x,y) = y^2$, or $\phi(x,y) = x^2 + y^2$

Example: With $\phi(x,y) = x + y - 2$, and the region R shown, calculate $\Phi = \iint_R \phi(x,y)\,dA$.



SOLUTION:

$$\Phi = \iint_R \phi(x,y)\,dA = \int_0^1 \int_{x^2}^{x} (x+y-2)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} - 2y\right]\Bigg|_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left[\left(x^2 + \frac{x^2}{2} - 2x\right) - \left(x^3 - \frac{x^4}{2} - 2x^2\right)\right] dx$$

$$= \int_0^1 \left[\frac{3}{2}x^2 - 2x - x^3 + \frac{x^4}{2} + 2x^2\right] dx = \int_0^1 \left(\frac{x^4}{2} - x^3 + \frac{7}{2}x^2 - 2x\right) dx$$

$$= \left[\frac{x^5}{10} - \frac{x^4}{4} + \frac{7}{6}x^3 - x^2\right]\Bigg|_0^1 = \frac{1}{10} - \frac{1}{4} + \frac{7}{6} - 1 = \frac{6 - 15 + 70 - 60}{60}$$

$$= \frac{76 - 75}{60} = \frac{1}{60}$$

To calculate double integrals in the HP 49G+ or TI 89, simply use the integral symbol twice entering the proper limits, e.g., for this case:

HP 49G+:

$$\int_0^1 \int_{x^2}^{x} (x+y-2)\,dy\,dx$$

[ENTER] [EVAL]

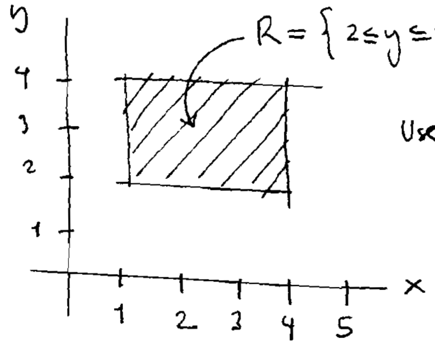
TI-89:

∫(∫(x+y−2, y, x^2, x), x, 0, 1) [ENTER]

Examples

RECTANGULAR INTEGRATION REGION

$R = \{2 \le y \le 4,\ 1 \le x \le 4\}$



Use $f(x,y) = x^2 + y^2$

$$\iint_R f(x,y)\,dA = \int_1^4 \int_2^4 (x^2+y^2)\,dy\,dx$$

$$= 98$$

or, $\displaystyle\int_2^4 \int_1^4 (x^2+y^2)\,dx\,dy = 98.$

EXAMPLE of Exchange of integration order.

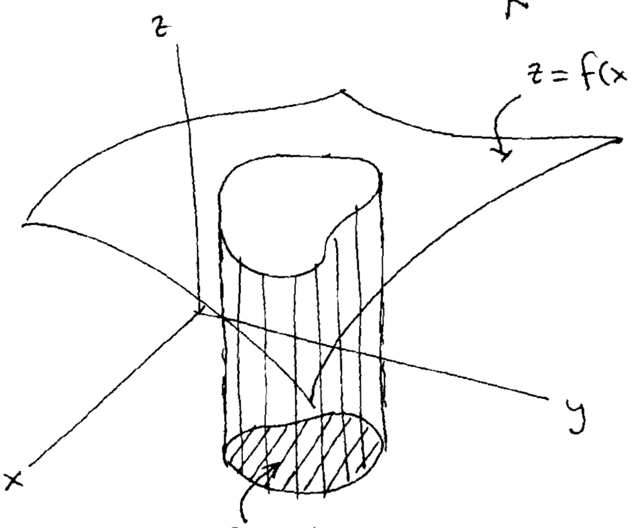
Use $f(x,y) = x^2\sqrt{y^2+y}$, so that $f(x,y) = h(x)g(y)$ with $h(x) = x^2$, $g(y) = \sqrt{y^2+y}$

$$\int_1^4 \int_2^4 x^2\sqrt{y^2+y}\,dy\,dx = \frac{21\ln(9-4\sqrt{5}) - (21\ln(5-2\sqrt{6}) - (756\sqrt{5} - 210\sqrt{6}))}{8} = 145.45$$

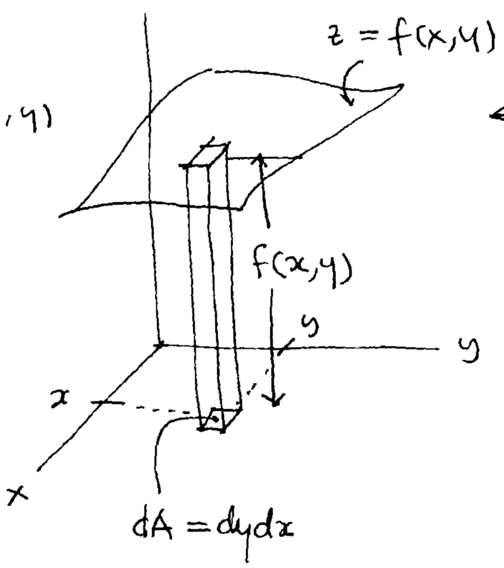
$$\left(\int_1^4 x^2\,dx\right)\left(\int_2^4 \sqrt{y^2+y}\,dx\right) =$$

$$= (21)\left(\frac{\ln(9-4\sqrt{5}) - (\ln(5-2\sqrt{6}) - (36\sqrt{5} - 10\sqrt{6}))}{8}\right) = 145.45$$

NOTE: INTERPRETATION of $\iint_R f(x,y)\,dA$. Function $z = f(x,y)$ represents a surface in space.



R = integration region in the x-y plane



← volume of the element with base at dA and height f(x,y):

$$dV = f(x,y)\,dA$$

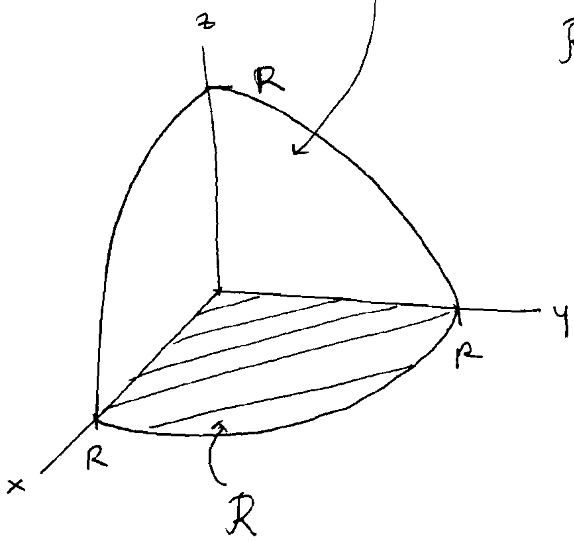
Thus, the volume of the body limited by region R and the surface $z = f(x,y)$ is calculated as the double integral of $f(x,y)$ over region R.

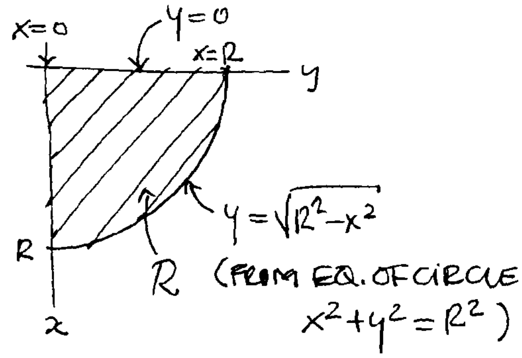
$$V = \iint_R dV = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dy\,dx$$

Example

$z = f(x,y) = \sqrt{R^2 - x^2 - y^2}$ (from eq. of sphere $x^2+y^2+z^2=R^2$)

$R = \{0 \le y \le \sqrt{R^2-x^2},\ 0 \le x \le R\}$

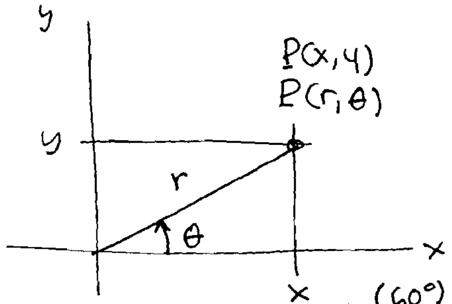




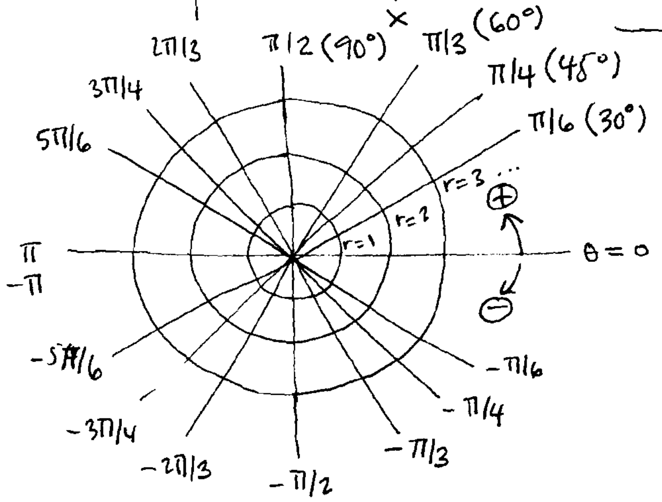
Thus, volume of 1/8 of a sphere, $V_{1/8} = \iint\limits_R f(x,y)\,dA = \int_0^R \int_0^{\sqrt{R^2-x^2}} \sqrt{R^2-x^2-y^2}\,dy\,dx = \frac{\pi R^3}{6}$

volume of a sphere, $V = 8V_{1/8} = 8 \cdot \frac{\pi R^3}{6} = \frac{4}{3}\pi R^3$

Area (double) integrals in polar coordinates



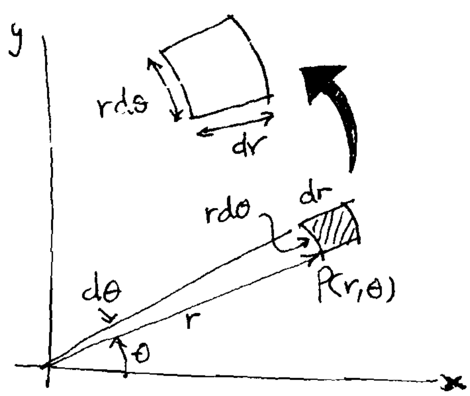
Converting from	to	use
Cartesian	Polar	$r=\sqrt{x^2+y^2}$ $\theta = \tan^{-1}(y/x)$
Polar	Cartesian	$x = r\cos\theta$ $y = r\sin\theta$



POLAR COORDINATE GRID

Integrals involving circular shapes can be easily calculated in polar coordinates

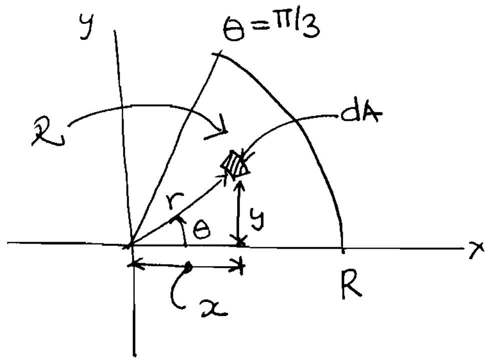
ELEMENT OF AREA in polar coordinates $\quad dA = r\,dr\,d\theta$



DOUBLE INTEGRALS in polar coordinates

$$\iint\limits_R f(r,\theta)\,dA = \iint\limits_R f(r,\theta)\,r\,dr\,d\theta$$

EXAMPLES



AREA, $A=\iint_R dA = \int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=R} r\,dr\,d\theta = \int_0^{\pi/3} \frac{r^2}{2}\Big|_0^R d\theta$

$= \int_0^{\pi/3}\frac{R^2}{2}\,d\theta = \frac{R^2}{2}\,\theta\Big|_0^{\pi/3} = \frac{\pi R^2}{6}$

$R=\{0\le r\le R,\ 0\le\theta\le\pi/3\}$

FIRST MOMENTS of area (recall, $x=r\cos\theta$, $y=r\sin\theta$)

$$M_x^A = \iint_R y\,dA = \int_0^{\pi/3}\int_0^R r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi/3}\int_0^R r^2\sin\theta\,dr\,d\theta$$

$$= \left[\int_0^{\pi/3}\sin\theta\,d\theta\right]\left[\int_0^R r^2\,dr\right] = \left[-\cos\theta\Big|_0^{\pi/3}\right]\left[\frac{r^3}{3}\Big|_0^R\right]$$

$$= (-\cos\pi/3+\cos 0)\left(\frac{R^3}{3}\right) = \left(-\frac{1}{2}+1\right)\frac{R^3}{3} = \frac{1}{2}\cdot\frac{R^3}{3} = \frac{R^3}{6}$$

$$M_y^A = \iint_R x\,dA = \int_0^{\pi/3}\int_0^R r\cos\theta\, r\,dr\,d\theta = \int_0^{\pi/3}\int_0^R r^2\cos\theta\,dr\,d\theta = \left(\int_0^{\pi/3}\cos\theta\,d\theta\right)\left(\int_0^R r^2\,dr\right)$$

$$= \left(\sin\theta\Big|_0^{\pi/3}\right)\left(\frac{R^3}{3}\right) = (\sin\pi/3-0)\frac{R^3}{3} = \left(\frac{\sqrt3}{2}\right)\left(\frac{R^3}{3}\right) = \frac{\sqrt3}{6}R^3$$

MOMENTS of INERTIA

$$I_x = \iint_R y^2\,dA = \int_0^{\pi/3}\int_0^R r^2\sin^2\theta\, r\,dr\,d\theta = \int_0^{\pi/3}\int_0^R r^3\sin^2\theta\,dr\,d\theta = \frac{4\pi-3\sqrt3}{96}R^4 = 7.68\times10^{-2}R^4$$

↑ calculated using HP 49G+

NOTE: to enter θ type $\boxed{\text{ALPHA}}\ \boxed{\urcorner}\ \boxed{\text{T}}$

$$I_y = \iint_R x^2\,dA = \int_0^{\pi/3}\int_0^R r^2\cos^2\theta\, r\,dr\,d\theta = \int_0^{\pi/3}\int_0^R r^3\cos^2\theta\,dr\,d\theta = \frac{4\pi+3\sqrt3}{96}R^4 = 0.185R^4$$

CENTROID:

$$x_c = \frac{M_y^A}{A} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\frac{\sqrt3}{6}R^3}{\frac{\pi R^2}{6}} = \frac{\sqrt3}{\pi}R,\quad y_c = \frac{M_x^A}{A} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{R^3/6}{\pi R^2/6} = \frac{R}{\pi}$$

in polar coordinates $r_c = \sqrt{x_c^2+y_c^2} = \sqrt{\frac{3}{\pi^2}R^2+\frac{R^2}{\pi^2}} = \sqrt{\frac{4R^2}{\pi^2}} = \frac{2R}{\pi}$

$$\theta_c = \tan^{-1}\left(\frac{y_c}{x_c}\right) = \tan^{-1}\left(\frac{R/\pi}{\sqrt3R/\pi}\right) = \tan^{-1}\left(\frac{\sqrt3}{3}\right) = \frac{\pi}{6}$$

radii of gyration

$$K_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{4\pi-3\sqrt3}{96}R^4}{\pi R^2/6}} = \sqrt{\frac{4\pi-3\sqrt3}{16}R^2} = \frac{\sqrt{4\pi-3\sqrt3}}{4}R = 0.679R$$

$$K_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{4\pi+3\sqrt3}{96}R^4}{\pi R^2/6}} = \sqrt{\frac{4\pi+3\sqrt3}{16}R^2} = \frac{\sqrt{4\pi+3\sqrt3}}{4}R = 1.053R$$

$$K_o = \sqrt{\frac{I_x+I_y}{A}} = \sqrt{\frac{\pi R^4/12}{\pi R^2/6}} = \frac{R^2}{2}$$

$$I_x+I_y = \left(\frac{4\pi-3\sqrt3}{96}+\frac{4\pi+3\sqrt3}{96}\right)R^4 = \frac{8\pi}{96}R^4 = \frac{\pi R^4}{12}$$