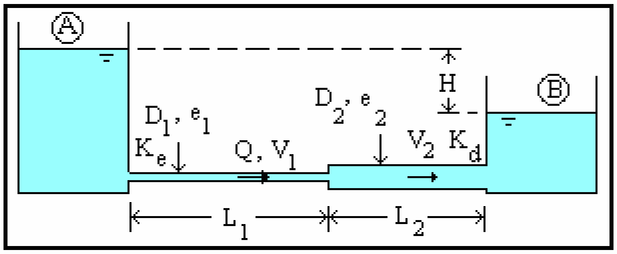
Pipes in series with expansion loss – solution using the HP and TI calculators
By Gilberto E. Urroz, March 2006

To illustrate the solution for the flow in a system of pipelines in series consider the case of two reservoirs connected by two pipelines in series so that the difference in elevation of the free surface in both reservoirs is *H*. This case assumes an expansion so that the diameters of the pipelines, as illustrated below, are $D_1<D_2$. The analysis of a contraction in the pipelines series problem would be slightly different than the present one.



If we refer to the upstream reservoir as *A* and the downstream reservoir as *B*, we can take $z_B = 0$ and $z_A = H$. At the free surfaces we have $p_A = p_B = 0$, and $V_A = V_B = 0$. Energy losses to consider are the friction losses in each pipeline, given by

$$h_{f1} = f_1 \frac{L_1}{D_1}\frac{V_1^2}{2g}, \quad h_{f2} = f_2 \frac{L_2}{D_2}\frac{V_2^2}{2g},$$

and minor (or local) losses at the pipeline entrance (sub index *e*), at the expansion between the two pipelines (sub index *x*), and at the pipe discharge (sub index *d*). These minor losses will be written as

$$h_{L(e)} = K_e \frac{V_1^2}{2g}, \quad h_{L(x)} = K_x \frac{V_1^2}{2g}, \quad h_{l(d)} = K_d \frac{V_2^2}{2g}.$$

The actual expression for the expansion head losses is as follows (using continuity, i.e., $V_1D_1 = V_2D_2$):

$$h_{L(x)} = \frac{(V_1 - V_2)^2}{2g} = \left[1 - \left(\frac{D_1}{D_2}\right)^2\right]^2 \frac{V_1^2}{2g}.$$

Thus,

$$K_x = \left[1 - \left(\frac{D_1}{D_2}\right)^2\right]^2 = \left(\frac{D_2^2 - D_1^2}{D_2^2}\right)^2.$$

The energy equation between free surfaces *A* and *B* is:

$$z_A + \frac{p_A}{\gamma} + \frac{V_A^2}{2g} = z_B + \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + \sum h_{fi} + \sum h_{Li} \ .$$

After introducing the values of elevations, velocities, pressures, and losses into this equation, the equation becomes:

$$H = f_1 \frac{L_1}{D_1}\frac{V_1^2}{2g} + f_2 \frac{L_2}{D_2}\frac{V_2^2}{2g} + K_e \frac{V_1^2}{2g} + K_x \frac{V_1^2}{2g} + K_d \frac{V_2^2}{2g} \, ,$$

or,

$$H = \frac{V_1^2}{2g}\left(f_1 \frac{L_1}{D_1} + K_e + K_x \right) + \frac{V_2^2}{2g}\left(f_2 \frac{L_2}{D_2} + K_d \right).$$

Using the following expressions for the velocities: $V_1 = \frac{4Q}{\pi D_1^2}, \quad V_2 = \frac{4Q}{\pi D_2^2}$,

and appropriate expressions for the Reynolds numbers, the equation becomes:

$$H = \frac{8Q^2}{\pi^2 g D_1^4}\left(f_1\left(\frac{e_1}{D_1}, \frac{4Q}{\pi \nu D_1}\right)\frac{L_1}{D_1} + K_e + K_x \right) + \frac{8Q^2}{\pi^2 g D_2^4}\left(f_2\left(\frac{e_2}{D_2}, \frac{4Q}{\pi \nu D_2}\right)\frac{L_2}{D_2} + K_d \right).$$

which can be re-arranged to read:

$$H = \frac{8Q^2}{\pi^2 g}\left(f_1\left(\frac{e_1}{D_1}, \frac{4Q}{\pi \nu D_1}\right)\frac{L_1}{D_1^5} + \frac{K_e + K_x}{D_1^4} + f_2\left(\frac{e_2}{D_2}, \frac{4Q}{\pi \nu D_2}\right)\frac{L_2}{D_2^5} + \frac{K_d}{D_2^4} \right).$$

The minor loss terms in the expression between the parentheses can be simplified somehow by replacing the expression for K_x as indicated next. Let K_S be defined as:

$$K_S = \frac{K_e + K_x}{D_1^4} + \frac{K_d}{D_2^4} = \frac{K_e}{D_1^4} + \frac{1}{D_1^4}\left(\frac{D_2^2 - D_1^2}{D_2^2}\right)^2 + \frac{K_d}{D_2^4} = \frac{K_e}{D_1^4} + \frac{K_d}{D_2^4} + \left(\frac{1}{D_1^2} - \frac{1}{D_2^2}\right)^2 .$$

Replacing this term in the equation for *H*, above, we get:

$$H = \frac{8Q^2}{\pi^2 g}\left(f_1\left(\frac{e_1}{D_1}, \frac{4Q}{\pi \nu D_1}\right)\frac{L_1}{D_1^5} + f_2\left(\frac{e_2}{D_2}, \frac{4Q}{\pi \nu D_2}\right)\frac{L_2}{D_2^5} + K_S \right). \qquad \text{[Eq (A)]}$$

Notice that the value of K_S is a constant value for a given set of diameters D_1 and D_2. The value of K_S needs to be calculated before solving for any other unknown in the equation (A). To calculate K_S use:

$$K_S = \frac{K_e}{D_1^4} + \frac{K_d}{D_2^4} + \left(\frac{1}{D_1^2} - \frac{1}{D_2^2}\right)^2 \quad \text{[Eq (B)].}$$

Equation (A) can be entered (with a lot of patience) into the *Eq* or *eqn* fields of the numerical solvers in the HP or TI calculators, respectively. The equation to enter in the HP calculator is:

```
H=8*Q^2/(π^2*g)*(DARCY(e1/D1,4*Q/(π*nu*D1))*L1/D1^5 +
DARCY(e2/D2,4*Q/(π*nu*D2))*L2/D2^5+KS)
```

While, for the TI calculator you may use:

```
h=8*q^2/(π^2*g)*(fha(e1/d1,4*q/(π*nu*d1))*l1/d1^5 +
fha(e2/d2,4*q/(π*nu*d2))*l2/d2^5+ks)
```

where *fha* is the function for calculating the *Haaland's* approximation to the *Darcy-Weisbach* friction factor, i.e.,

$$f = \frac{0.3086}{\log^2\left[\left(\frac{e}{3.7\cdot D}\right)^{1.11} + \frac{6.9}{\mathbf{R}}\right]}.$$

If you haven't programmed function *fha* in your TI 89 calculator (from a previous exercise), proceed as follows:

1 - Select the *Program Editor* in your calculator.
2 - Select the option *3:New* to enter a new function.
3 - Select:

Type:	*2:Function*
Folder:	*(your favorite folder, e.g., one called "fluids")*
Variable:	*fha*

and press [Enter].
4 - Edit the function such that the editor's window looks as follows:

```
:fha(k,r)
:Func
:.3086/(log((k/3.7)^1.11+6.9/r)^2)
:EndFunc
```

Return to the *HOME* screen.

Example 1. Calculating H. As an example, use the following data to solve a series pipeline problem:

$$Q = 0.1\ m^3/s,\ \nu = 1\times 10^{-6}\ m^2/s,\ D_1 = 0.15\ m,\ D_2 = 0.30\ m,\ L_1 = 50\ m,$$
$$L_2 = 160\ m,\ e_1 = e_2 = 0.10\times 10^{-3}\ m,\ K_e = 0.5,\ K_d = 1.0, \text{ and } g = 9.806\ m/s^2.$$

The problem consists in finding H. The value of K_S is calculated from Eq. (B), as

$$K_S = \frac{K_e}{D_1^4} + \frac{K_d}{D_2^4} + \left(\frac{1}{D_1^2} - \frac{1}{D_2^2}\right)^2 = \frac{0.5}{0.15^4} + \frac{1.0}{0.30^4} + \left(\frac{1}{0.15^2} - \frac{1}{0.30^2}\right)^2 = 2222.22/m^4$$

for this case. A numerical solution using an HP 49G+ calculator produces $H = 12.72\ m$, while a numerical solution using a TI 89 calculator (with the function *sj(k,r)* programmed in it), produces $H = 12.69\ m$.

Example 2. Calculating Q. As a second example, assume that for the same system of pipelines and reservoirs you want to find the discharge that will occur if the difference in elevations between the two reservoirs is $H = 8.0\ m$. Changing the value of H in the numerical solutions, and solving for Q produces $Q = 0.079\ m^3/s$ in both the HP calculator and the TI calculators.

Example 3. Calculating D_2. Now, suppose that you want to keep the same data as in the original problem, except that you want to change the diameter D_2 in order to carry a flow of $Q = 0.2\ m^3/s$ with $H = 60\ m$. What would the value of D_2 be?

Since the value of D_2 affects the value of K_S [Eq(B)], this solution will have to be done in an iterative way:

1. Assume a value of $D_2 > D_1$.
2. Calculate K_S from Eq. (B) [re-written as Eq. (B') below]
3. Calculate D_2 from Eq. (A), i.e., using the numerical solver in the calculator
4. Repeat steps 2 and 3 until the values of D_2 in step 3 converge to a solution

The value of K_S for this case can be calculated from

$$K_S = \frac{K_e}{D_1^4} + \frac{K_d}{D_2^4} + \left(\frac{1}{D_1^2} - \frac{1}{D_2^2}\right)^2 = \frac{0.5}{0.15^4} + \frac{1.0}{D_2^4} + \left(\frac{1}{0.15^2} - \frac{1}{D_2^2}\right)^2 \qquad \text{[Eq (B')]}$$

The calculations are summarized in the following table:

HP 49G+ calculator			TI 89 calculator		
D_2(m)	K_S	new D_2(m)	D_2(m)	K_S	new D_2(m)
0.4	2485.53	0.2345	0.4	2485.53	0.2343
0.2345	2007.87	0.2291	0.2343	2007.41	0.229
0.2291	1995.43	0.229	0.229	1995.4	0.2289
0.229	1995.16	0.229	0.2289	1994.98	0.2289

The result is $D_2 = 0.229\ m$ in the HP calculator, and $D_2 = 0.2289\ m$ in the TI calculator. You may want to select a round value of $D_2 = 0.25\ m$, which would produce (check it out) $K_S = 2052.74/m^4$, and $H = 55.07\ m$ in the HP calculator, and $K_S = 2052.74/m^4$, and $H = 55.05\ m$ in the TI calculator.

Non-iterative solver solution. To avoid the iterative solution it is possible to define a function

$$K_S(K_e,K_d,D_1,D_2) = K_S = \frac{K_e}{D_1^4} + \frac{K_d}{D_2^4} + \left(\frac{1}{D_1^2} - \frac{1}{D_2^2}\right)^2, \qquad \text{[Eq. (B”)]}$$

in your calculator, and re-write Eq.(A) as:

$$H = \frac{8Q^2}{\pi^2 g}\left(f_1\left(\frac{e_1}{D_1}, \frac{4Q}{\pi\nu D_1}\right)\frac{L_1}{D_1^5} + f_2\left(\frac{e_2}{D_2}, \frac{4Q}{\pi\nu D_2}\right)\frac{L_2}{D_2^5} + K_S(K_e, K_d, D_1, D_2)\right). \qquad \text{[Eq (A’)]}$$

To define function K_S in the *HP 48/49/49G+* calculators use:

`‘KS(KE,KD,D1,D2)=KE/D1^4+KD/D2^4+(1/D1^2+1/D2^2)^2’` [ENTER][↰][DEF]

To define function K_S in the *TI 89* calculator use these steps:

1 - Select the *Program Editor* in your calculator.
2 - Select the option *3:New* to enter a new function.
3 - Select:

Type: *2:Function*
Folder: *(your favorite folder, e.g., one called “fluids”)*
Variable: *fha*

and press [Enter].
4 - Edit the function such that the editor’s window looks as follows:

```
:ks(ke,kd,d1,d2)
:Func
:ke/d1^4+kd/d2^4+(1/d1^2-1
 /d2^2)^2
:EndFunc
```

Return to the *HOME* screen.

Eq. (A') is entered as follows in the HP calculator:

```
H=8*Q^2/(π^2*g)*(DARCY(e1/D1,4*Q/(π*nu*D1))*L1/D1^5 +
DARCY(e2/D2,4*Q/(π*nu*D2))*L2/D2^5+KS(KE,KD,D1,D2))
```

while, for the TI calculator you may use:

```
h=8*q^2/(π^2*g)*(fha(e1/d1,4*q/(π*nu*d1))*l1/d1^5 +
fha(e2/d2,4*q/(π*nu*d2))*l2/d2^5+ks(ke,kd,d1,d2))
```

Using the solver in any of the two calculators will produce entries for the coefficients K_e, K_d, instead of the coefficient K_s. With this formulation it is possible to solve directly for D_2. The result found for both the *hp 49g+* and *TI 89* calculators is $D_2 = 0.2289\ m$. If selecting $D_2 = 0.25\ m$ for the final design, the head loss becomes $H = 55.07\ m$ for the *hp 49g+* calculator, and $H = 55.05$ for the *TI 89* calculator.

Exercise. For a system of a two pipes in series with $D_1<D_2$, $\nu = 1\times 10^{-6}\ m^2/s$, $e_1 = e_2 = 0.10 \times 10^{-3}\ m$, $K_e = 0.65$, $K_d = 1.0$, and $g = 9.806\ m/s^2$. Solve for the missing variable in the table:

H(m)	Q(m^3/s)	D_1(m)	L_1(m)	D_2(m)	L_2(m)
?	0.035	0.1	1200	0.2	1400
300	?	0.2	1200	0.35	1400
1.5	0.05	?	300	0.4	500
15	0.15	0.3	?	0.5	200
4	0.1	0.25	300	?	300
10	0.6	0.25	1500	0.4	?