

NOTE: In the set of lectures 19/20 I defined the length ratio as

$$L_r = L_m/L_p$$

The textbook by Finnermore & Franzini defines it as

$$L_r = L_p/L_m$$

To avoid confusion let's keep the textbook definition, although, either way is fine as long as we keep in mind which definition we are using, i.e., *model/prototype* or *prototype/model*.

Dimensional Analysis

- Mathematical technique – study of dimensions
- Steps:
 1. Select physical parameters in a flow (or other phenomenon)
 2. Group parameters into dimensionless combination
- Useful in the design of experiments
- Express physical quantities using one of 2 systems of dimensions:
 1. F, L, T (force, length, time)
 2. M, L, T (mass, length, time)

These two systems are related (dimensionally) by Newton's second law: $F = MLT^{-2}$

Basic concepts:

- Principle of dimensional homogeneity (PDH) – Fourier 1822
- All theoretical equations that relate physical quantities must be dimensionally homogeneous, i.e., both sides of the equation must have the same dimensions.
- Example: power developed by a pump: $P = Q \gamma h$

$$[P] = FLT^{-1}, [Q] = L^3T^{-1}, [\gamma] = FL^{-3}, [h_p] = L$$

$$[Q \gamma h_p] = L^3T^{-1} \cdot FL^{-3} \cdot L = FLT^{-1}$$

$$\Rightarrow [P] = [Q \gamma h_p]$$

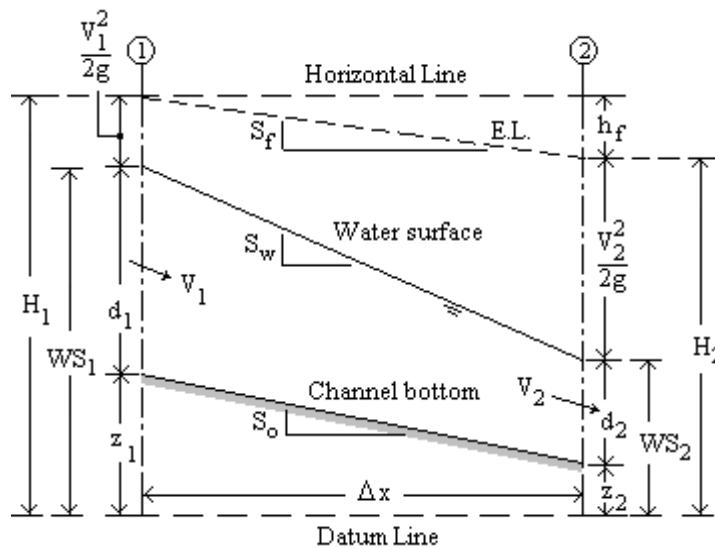
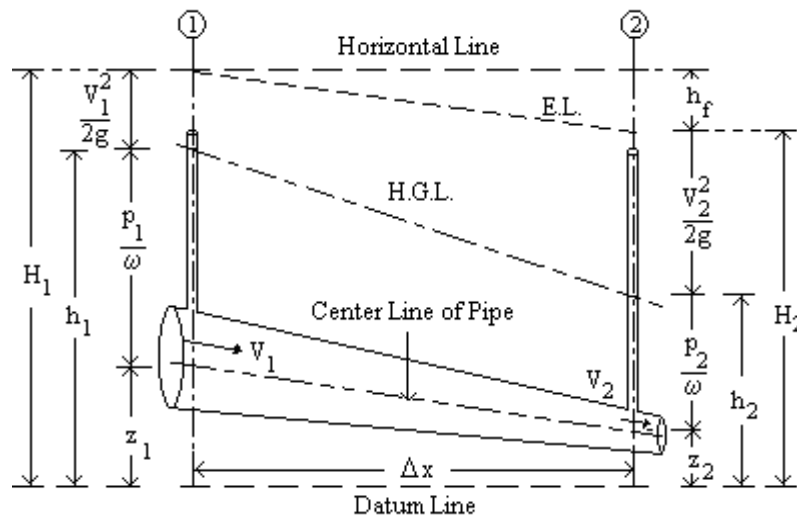
- NOTE: I prefer this notation over the textbook's notation, e.g., $P = [FLT^{-1}]$

- Most famous non-homogeneous equation used in fluid mechanics and hydraulics: Manning's equation (used in pipe flow and open channel flow):

$$\text{BG Units: } V = \frac{1.486}{n} \cdot R_h^{2/3} \cdot \sqrt{S} \quad \text{SI Units: } V = \frac{1}{n} \cdot R_h^{2/3} \cdot \sqrt{S}$$

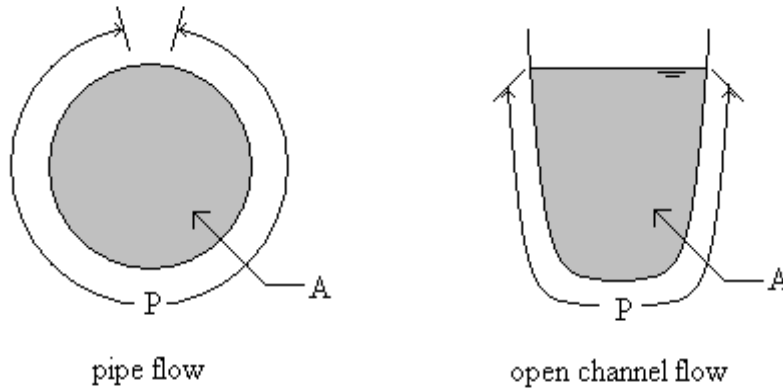
n = roughness coefficient, e.g., for concrete lining, $n = 0.012$ (dimensionless quantity)
 V = flow velocity (LT^{-1}), $R_h = A/P$ = hydraulic radius (L), A = area (L^2), P = wetted perimeter (L), $S = h_f/L$ = energy line gradient (dimensionless), h_f = friction head losses (L), L = pipe or channel length (L).

Energy heads in pipe and open channel flow are illustrated in the following figures:



NOTE: The figures illustrate non-uniform flow in pipes and open channels (i.e., a pipe of varying diameter, and varying depth in an open channel). Manning's equation is valid only for uniform flow, i.e., a pipe of constant diameter or a uniform flow (constant depth) in open channel flow.

The cross-sectional geometric elements are illustrated in the figure below:



For pipe flow, $A = \pi D^2/4$ and $P = \pi D$, and $R = A/P = D/4$. For an open channel, $A = A(y)$, and $P = P(y)$, thus, $R_h = A/P = R_h(y)$.

Manning's equation can be written as a single expression for both systems of units (SI, English) as

$$V = \frac{C_u}{n} \cdot R_h^{2/3} \cdot \sqrt{S}$$

if we specify $C_u = 1.000$ for *SI*, and $C_u = 1.486$ for *ES*. The values of C_u and n are typically taken as dimensionless, while S is dimensionless by definition ($S = h_f/L$). Thus,

$$[C_u] = 1, [n] = 1, [S] = 1.$$

The dimension of the other variables involved in the equation are:

$$[V] = \text{LT}^{-1}, [R_h] = \text{L}.$$

- The left-hand side of the equation has units of velocity, $[V] = \text{LT}^{-1}$
- The right-hand side of the equation has units of

$$\left[\frac{C_u}{n} \cdot R_h^{2/3} \cdot \sqrt{S} \right] = \frac{1}{1} \cdot \text{L}^{2/3} \sqrt{1} = \text{L}^{2/3}$$

Thus, this equation is clearly dimensionally non-homogeneous.

How do we make Manning's equation homogeneous?

1 – Keep C_u dimensionless and give n dimensions of

$$[n] = \left[\frac{C_u}{V} \cdot R_h^{2/3} \cdot \sqrt{S} \right] = \frac{1}{L \cdot T^{-1}} \cdot L^{2/3} \cdot \sqrt{1} = L^{-1/3} \cdot T \quad ,$$

i.e., the units of n would be $s/m^{1/3}$ (SI) or $s/ft^{1/3}$ (ES). However, the values of n are given in the literature as a single value (that depends on the lining material only), regardless of the system of units used. Therefore, this approach would not be practical.

2 – Keep n dimensionless and give C_u dimensions of

$$[C_u] = \left[\frac{n \cdot V}{R_h^{2/3} \cdot \sqrt{S}} \right] = \frac{1 \cdot L \cdot T^{-1}}{L^{2/3} \cdot 1^{1/2}} = L^{1/3} \cdot T^{-1} \quad ,$$

i.e., the value of C_u would be $C_u = 1.0 \text{ m}^{1/3}/\text{s}$ (SI) or $C_u = 1.486 \text{ ft}^{1/3}/\text{s}$ (ES).

With $[C_u] = L^{1/3}T^{-1}$, the right-hand side of Manning's equation now has units of velocity:

$$\left[\frac{C_u}{n} \cdot R_h^{2/3} \cdot \sqrt{S} \right] = \frac{L^{1/3} \cdot T^{-1}}{1} \cdot L^{2/3} \sqrt{1} = L \cdot T^{-1}$$

and the equation would be dimensionally homogeneous.

Attaching units to the coefficient C_u is useful, for example, when using units in solving Manning's equations problems in *Mathcad*, as illustrated in the example shown in the following page.

3 – For flow in rivers, research suggests that the Manning's resistance coefficient n is related to the mean diameter of bed sediments by an equation of the form $n = kd^{1/6}$. With this result, and if k is taken as dimensionless, $[n] = L^{1/6}$. From the Manning's equation, the dimensions of C_u would be

$$[C_u] = \left[\frac{n \cdot V}{R_h^{2/3} \cdot S^{1/2}} \right] = \frac{L^{1/6} \cdot L \cdot T^{-1}}{L^{2/3} \cdot 1^{1/2}} = L^{1/2} \cdot T^{-1} \quad ,$$

which are the dimensions of the square root of an acceleration $\sqrt{L \cdot T^{-2}} = L \cdot T^{-1}$. Thus, we would have to specify $C_u = 1.0 \text{ m}^{1/2}/\text{s}$ (SI) or $C_u = 1.486 \text{ ft}^{1/2}/\text{s}$ (ES).

The problem with this approach is that Manning's n would have to have units of $m^{1/6}$ or $ft^{1/6}$ attached to its value (e.g., for concrete lining, $n = 0.012 \text{ m}^{1/2}/\text{s}$ or is it $n = 0.012 \text{ ft}^{1/2}/\text{s}$?). This approach would require a re-writing of the extensive data base of Manning's numbers in dimensional terms, which makes it again not very practical.

Of all the three approaches presented above, therefore, approach number 2 would be the most practical to implement with the existing databases of values of n .

Example of solution of Manning's equation using Mathcad – Notice that the coefficient C_u is given units of $L^{1/3}T^{-1}$.

Consider a trapezoidal channel of bottom width $b = 4.5$ ft, and $z = 1.5$, carrying a discharge $Q = 20$ cfs, with $n = 0.012$, and $S_o = 0.00023$. Determine the flow depth (this is referred to as the normal depth of flow). The solution is shown next:



$$Q := 20 \frac{\text{ft}^3}{\text{sec}} \quad b := 4.5 \text{ ft} \quad z := 1.5 \quad n := 0.012$$

$$S_o := 0.00023 \quad C_u := 1.486 \text{ ft}^{\frac{1}{3}} \cdot \text{sec}^{-1}$$

Initial guess: $y := 2.5 \text{ ft}$

Given

$$Q = \frac{C_u}{n} \cdot \frac{[(b + z \cdot y) \cdot y]^{\frac{5}{3}} \cdot \sqrt{S_o}}{\left(b + 2 \cdot y \cdot \sqrt{1 + z^2} \right)^{\frac{2}{3}}} \quad +$$

$$y_n := \text{Find}(y)$$

Solution: $y = 0.415 \text{ ft}$

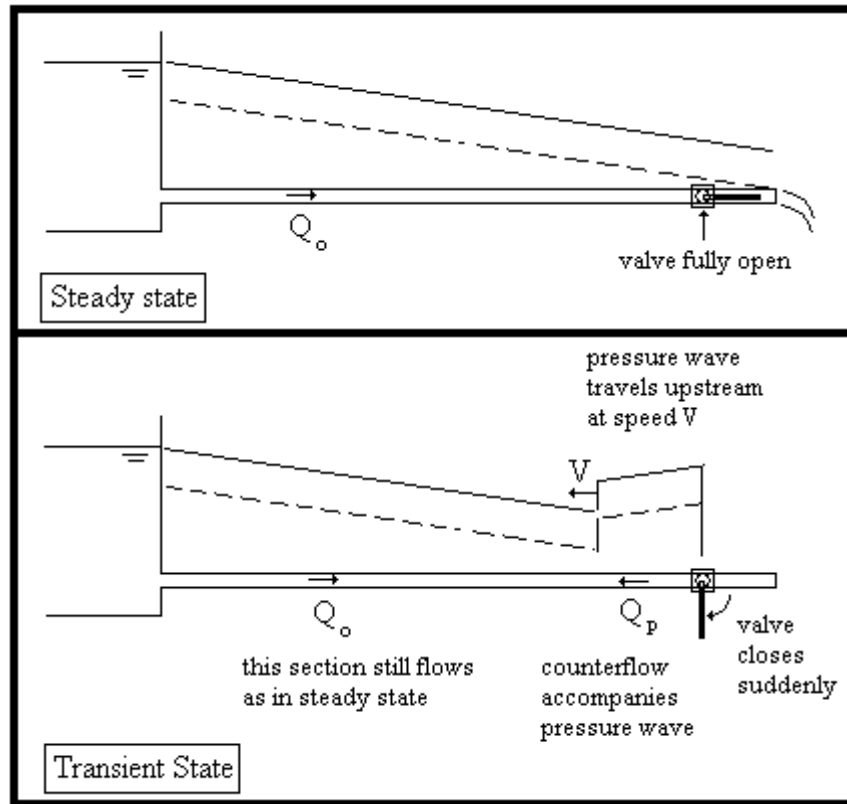
Check result:

$$\frac{C_u}{n} \cdot \frac{[(b + z \cdot y) \cdot y]^{\frac{5}{3}} \cdot \sqrt{S_o}}{\left(b + 2 \cdot y \cdot \sqrt{1 + z^2} \right)^{\frac{2}{3}}} = 20 \frac{\text{ft}^3}{\text{s}}$$

The current engineering practice in relation to Manning's equation is to use a consistent set of units and the proper value of C_u , i.e., SI = {m,s, $C_u = 1$ } or ES = {ft, s, $C_u = 1.486$ }. In spite of being inherently non-homogeneous, Manning's equation has been used successfully in the analysis of open channel flow since it was developed by Manning, in Ireland, in the late 1800's.

Example of dimensional analysis in the design of experiments

Consider the case of a pressure wave in a pipe as illustrated in the figure below.



Originally, water in the pipe flows at a steady-state discharge Q_o while the valve near the pipe outlet is fully open. If the valve is suddenly closed, the kinetic energy of the flow is converted into pressure energy and the local pressure suddenly increases creating a pressure wave that travels upstream towards the reservoir at a speed V . While water at the upstream sections of the pipe still flow at the steady-state discharge, Q_o , a counterflow Q_p accompanies the pressure wave as illustrated in the figure. Eventually, the pressure wave reaches the reservoir, increasing the local pressure there, and reflecting back into the pipe. The pressure wave will travel back-and-forth in the pipe until friction dissipates the pressure energy and the flow through the pipe stops completely. The flow in the pipe after the valve closure constitutes a transient (time-dependent) condition known as the *water hammer*.

NOTES:

1 – The water hammer phenomenon is described in detail in pages 558-573 in the textbook by Finnemore and Franzini.

2 - Sometimes, you can hear a water hammer a a thumping noise when you close a faucet at your home, particularly if you close the valve very fast.

What variables would be involved in the mathematical description of the water hammer phenomenon?

- Wave speed, $[V] = \text{LT}^{-1}$
- Bulk modulus of elasticity, $[E] = \text{FL}^{-2} = \text{ML}^{-1}\text{T}^{-2}$
- Fluid density, $[\rho] = \text{ML}^{-3}$
- Kinematic viscosity, $[\nu] = \text{L}^2\text{T}^{-1}$

We propose a simple equation to model the relationship between the different variables involved:

$$V = C E^a \rho^b \nu^c$$

where C , a , b , and c are dimensionless constant values.

Dimensional homogeneity requires that the dimensions of both sides of the equation be the same, i.e.,

$$[V] = [C][E]^a[\rho]^b[\nu]^c$$

Replacing the dimensions of the variables involved, with $[C] = 1$ (dimensionless), we find:

$$\text{LT}^{-1} = 1 (\text{ML}^{-1}\text{T}^{-2})^a (\text{ML}^{-3})^b (\text{L}^2\text{T}^{-1})^c$$

i.e.,

$$\text{M}^0\text{L}\text{T}^{-1} = \text{M}^{a+b} \text{L}^{-a-3b+2c} \text{T}^{-2a-c}$$

Equating the exponents of the fundamental dimensions in both sides of the equations we end up with a system of three linear equations with three unknowns:

$$\text{M: } 0 = a+b \tag{1}$$

$$\text{L: } 1 = -a-3b+2c \tag{2}$$

$$\text{T: } -1 = -2a-c \tag{3}$$

This system can be written as a matrix equation, $\mathbf{Ax} = \boldsymbol{\beta}$, with

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -3 & 2 \\ -2 & 0 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

The solution to the system is $\mathbf{x} = \mathbf{A}^{-1}\boldsymbol{\beta}$. Using, for example, an *hp 49g+* calculator, we get

$$\mathbf{A}^{-1} = \begin{bmatrix} -3/2 & -1/2 & -1 \\ 5/2 & 1/2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

and,

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \boldsymbol{\beta} = \begin{bmatrix} -3/2 & -1/2 & -1 \\ 5/2 & 1/2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}.$$

The solution is, therefore,

$$a = 1/2, b = -1/2, c = 0,$$

or,

$$V = C E^a \rho^b v^c = C E^{1/2} \rho^{-1/2} v^c = C \cdot \sqrt{\frac{E}{\rho}}$$

Experiments would show that, in an infinite medium, $C = 1$ and $V = c = \sqrt{\frac{E}{\rho}}$ = speed of sound in water. [The symbol c is sometimes used in water hammer analysis for *celerity* of the pressure wave].

To determine the speed of sound in water we can use data from Table A4 in the textbook. For $T = 20^\circ\text{C}$, the speed of sound in water is shown below:

	$E(\text{N/m}^2) \times 10^6$	$\rho(\text{kg/m}^3)$	$c(\text{m/s})$	$c(\text{ft/s})$
Fresh water	2171	998	1475	4838
Sea water	2300	1500	1500	4920

In a pipeline the value of $V = c$ would also be affected by the elasticity of the pipe material as well as the type of anchoring of the pipe. Thus, the values of c shown in the table above are larger than the typical celerity waves in pipes. A typical value of c in pipes is $c = 1200 \text{ m/s} = 3936 \text{ ft/s}$.

The Pi Theorem (Buckingham Pi theorem)

$\Pi (P_i) = \text{dimensionless group of variables}$

Let $X_1, X_2, \dots, X_n = n$ dimensional variables (e.g., V, g, μ , etc.)

Involved in a physical phenomenon. Let

$$f(X_1, X_2, \dots, X_n) = 0 \text{ equation}$$

be the dimensionally homogeneous relating those variables.

We can re-arrange this equation into

$$\phi(\Pi_1, \Pi_2, \dots, \Pi_{n-k}) = 0$$

where ϕ is another function and each Π is an independent dimensionless product of some of the dimensional variables X_i 's.

$$k \leq m$$

There are $(n-k)$ Π terms, with ~~where~~, where $m = \text{number of fundamental dimensions involved in all the variables } X$.
In most cases, $k = m$.

ILLUSTRATIVE

EXAMPLE:

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

← position in uniformly-accelerated motion

Divide by x_0 :

$$\frac{x}{x_0} = 1 + \frac{v_0 t}{x_0} + \frac{a t^2}{2 x_0}$$

$$X_1 = x, X_2 = x_0$$

$$X_3 = v_0, X_4 = a$$

$$X_5 = t \Rightarrow n=5$$

$$[x] = L$$

$$[x_0] = L$$

$$[v_0] = LT^{-1}$$

$$[a] = LT^{-2}$$

$$[t] = T$$

$$\Pi_1 = 1 + \Pi_2 + \Pi_3$$

$$\Pi_1 = x/x_0, \Pi_2 = v_0 t/x_0, \Pi_3 = a t^2/2 x_0$$

$$n-k=3, \text{ with } n=5 \Rightarrow k=n-3=5-3=2 \leftarrow$$

Typically, ~~usually~~

number of dimensions involved = L, T

ANOTHER

EXAMPLE:

→ momentum equation: $F = \rho Q V_2 - \rho Q V_1$

$$X_1 = F, X_2 = \rho, X_3 = Q, X_4 = V_1, X_5 = V_2 \Rightarrow n=5$$

Divide by $\rho Q V_1$:

$$\frac{F}{\rho Q V_1} = \left(\frac{V_2}{V_1}\right) - 1, \text{ take } \Pi_1 = \frac{F}{\rho Q V_1}, \Pi_2 = \frac{V_2}{V_1}$$

$$[F] = MLT^{-2}$$

$$[\rho] = ML^{-3}$$

$$[V_1, V_2] = LT^{-1}$$

$$[Q] = L^3 T^{-1}$$

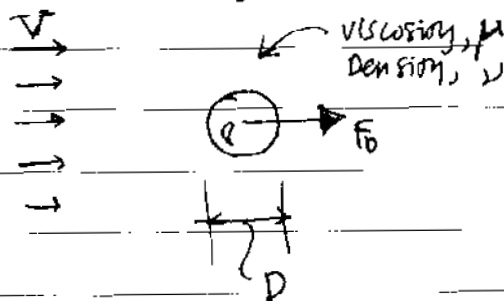
$$n-k=2 \Rightarrow k=5-2=3 \text{ dimensions}$$

$$MLT$$

Implementation of Pi Theorem

- ① List the n variables involved
- ② choose dimensional system (MLT or FLT)
- ③ Find the reduction k = number of basic dimensions needed
- ④ $(n-k)$ = Number of Π terms needed
- ⑤ select k repeating variables - if $k=3$, for example, select one mass (dynamic) variable, one geometric, and one kinematic variable, and form Π groups by multiplying the repeating variables raised to an exponent times the non-repeating variables one at a time.
- ⑥ Equate exponents of basic dimensions in each dimensionless group to form $(n-k)$ equations. Solve for the exponents.
- ⑦ Rearrange Π groups as desired

EXAMPLE: Drag force on a sphere



① $f(F_D, D, V, \rho, \mu) = 0, n=5$

② choose MLT

$$[F_D] = MLT^{-2} \quad [D] = L$$

$$[V] = LT^{-1} \quad [\rho] = ML^{-3}$$

$$[\mu] = ML^{-1}T^{-1}$$

③ $k=3$, since M, L, T used

④ number of Π terms needed: $n-k = 5-3 = 2$

⑤ select repeating variables: ρ (dynamic), D (geom), V (kinematic)

⑥ $\Pi_1 = \rho^{a_1} D^{b_1} V^{c_1} \mu, \quad \Pi_2 = \rho^{a_2} D^{b_2} V^{c_2} F_D$

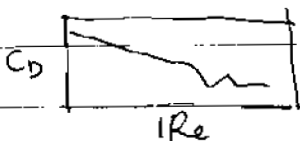
$$M^0 L^0 T^0 = M^{a_1} L^{-3a_1} L^{b_1} L^{c_1} T^{-c_1} M L^{-1} T^{-1} \Rightarrow \begin{cases} a_1 + 1 = 0 \\ -3a_1 + b_1 + c_1 - 1 = 0 \\ -c_1 - 1 = 0 \end{cases}$$

$$a_1 = -1, c_1 = -1, b_1 = -1 \Rightarrow \boxed{\Pi_1 = \mu / \rho D V}$$

For Π_2 : $M^0 L^0 T^0 = M^{a_2} L^{-3a_2} L^{b_2} L^{c_2} T^{-c_2} M L T^{-2} \Rightarrow \begin{cases} a_2 + 1 = 0 \\ -3a_2 + b_2 + c_2 + 1 = 0 \\ -c_2 - 2 = 0 \end{cases}$

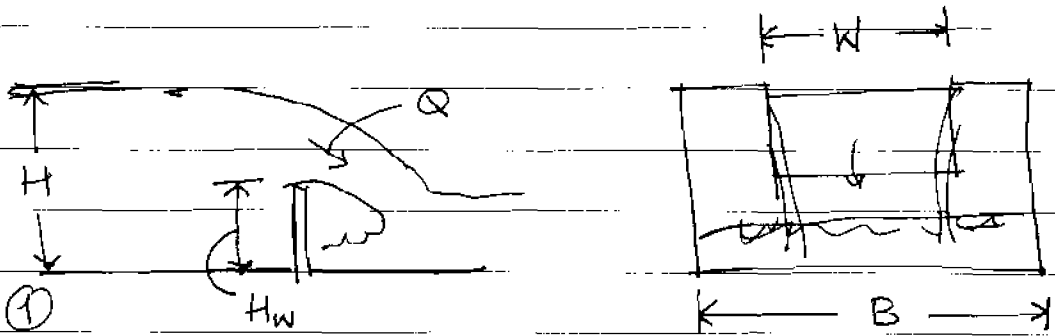
$$a_2 = -1, c_2 = -2, b_2 = -2 \Rightarrow \boxed{\Pi_2 = F_D / \rho D^2 V^2}$$

⑦ $f(F_D, D, V, \rho, \mu) = 0$ becomes $\phi_1(\Pi_1, \Pi_2) = 0$ or $\Pi_2 = \phi(\Pi_1^{-1})$
 because $\Pi_1^{-1} = \rho D V / \mu = Re \Rightarrow F_D / \rho D^2 V^2 = \phi(Re)$. Actually, in practice we'll write $2\Pi_2 = \phi(\Pi_1^{-1}) \Rightarrow \frac{F_D}{\frac{1}{2} \rho V^2} = \phi(Re)$



$$F_D = \frac{1}{2} \rho V^2 C_D, \text{ Drag coefficient } C_D = \phi(Re)$$

EXAMPLE - flow over a weir



W = width of weir

H = approach depth of flow

B = width of channel

Hw = weir height

Q = discharge

g = gravity ← driving force

σ = surface tension

← since it involves a free surface

$n = 8$

ρ = density

② use MLT ⇒ [W, H, Hw, B] = L, [Q] = LT⁻³, [g] = LT⁻²

[σ] = FL⁻¹ = MLT⁻²L⁻¹ = MT⁻²

[ρ] = ML⁻³

③ $K = 3$, number of dimensions involved

④ $n - k = 8 - 3 = 5$ Π terms needed

⑤ select repeating variables:

- dynamic (mass) → ρ

- geometric (length) → Hw

- kinematic (L/T) → g

form Π groups:

$$\begin{aligned} \Pi_1 &= \rho^{x_1} H_w^{y_1} g^{z_1} W = W/H_w \\ \Pi_2 &= \rho^{x_2} H_w^{y_2} g^{z_2} H = H/H_w \\ \Pi_3 &= \rho^{x_3} H_w^{y_3} g^{z_3} B = B/H_w \\ \Pi_4 &= \rho^{x_4} H_w^{y_4} g^{z_4} Q \\ \Pi_5 &= \rho^{x_5} H_w^{y_5} g^{z_5} \sigma \end{aligned} \left. \begin{array}{l} \text{by} \\ \text{inspection} \\ \text{(all length ratios)} \end{array} \right\}$$

⑥ $\Pi_4 \rightarrow M^0 L^0 T^0 = M^{x_4} L^{-3x_4} L^{y_4} L^{z_4} T^{-2z_4} L T^{-3} \Rightarrow \begin{cases} x_4 = 0 \\ -3x_4 + y_4 + z_4 + 3 = 0 \\ -2z_4 - 1 = 0 \end{cases}$

$x_4 = 0, z_4 = -1/2, 0 + y_4 - 1/2 + 3 = 0 \Rightarrow y_4 = -5/2$

$\Pi_4 = \rho^0 H_w^{-5/2} g^{-1/2} Q = Q / \sqrt{g H_w^5}$

$\Pi_5 \rightarrow M^0 L^0 T^0 = M^{x_5} L^{-3x_5} L^{y_5} L^{z_5} T^{-2z_5} M T^{-2} \Rightarrow \begin{cases} x_5 + 1 = 0 \\ -3x_5 + y_5 + z_5 = 0 \\ -2z_5 - 2 = 0 \end{cases}$

$x_5 = -1, z_5 = -1, 3 + y_5 - 1 = 0 \Rightarrow y_5 = -2$

$\Pi_5 = \rho^{-1} H_w^{-2} g^{-1} \sigma = \sigma / \rho g H_w^2 \Rightarrow \text{⑦ } \phi \left(\frac{Q}{\sqrt{g H_w^5}}, \frac{\sigma}{\rho g H_w^2}, \frac{W}{H_w}, \frac{H}{H_w}, \frac{B}{H_w} \right)$

IMPLEMENTATION OF DIMENSIONAL ANALYSIS WITH MATRICES

Drag force on a sphere

choose MLT

① $F(F_D, D, V, \rho, \mu) = 0, n=5$

② $[F_D] = MLT^{-2} \quad [D] = L$

③ $K=3$

$[V] = LT^{-1} \quad [\rho] = ML^{-3}$

④ Π terms: $n - k = 2$

⑤ repeating variables: ρ, D, V

$$\Pi_1 = \rho^{x_1} D^{y_1} V^{z_1} \mu$$

$$\Pi_2 = \rho^{x_2} D^{y_2} V^{z_2} F_D$$

$$\begin{matrix} M: \\ L: \\ T: \end{matrix} \left\{ \begin{array}{l} x_1 + 1 = 0 \\ -3x_1 + y_1 + z_1 - 1 = 0 \\ -z_1 - 1 = 0 \end{array} \right\} \quad \left\{ \begin{array}{l} x_2 + 1 = 0 \\ -3x_2 + y_2 + z_2 + 1 = 0 \\ -z_2 - 2 = 0 \end{array} \right\}$$

⇓

⇓

$$\left\{ \begin{array}{l} x_1 = -1 \\ -3x_1 + y_1 + z_1 = 1 \\ -z_1 = -1 \end{array} \right\} \quad \left\{ \begin{array}{l} x_2 = -1 \\ -3x_2 + y_2 + z_2 = -1 \\ -z_2 = 2 \end{array} \right\}$$

⇓

⇓

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{matrix} & \rho & D & V & & \mu & F_D \\ M: & [1 & 0 & 0] & [x_1 & x_2] & [-1 & -1] \\ L: & [-3 & 1 & 1] & [y_1 & y_2] & [1 & -1] \\ T: & [0 & 0 & -1] & [z_1 & z_2] & [-1 & 2] \end{matrix}$$

A

X

B

COEFFICIENTS OF M, L, T for repeating variables: each column corresponds to the M, L, T exponents of the repeating variables

↑
exponents of Π terms (by columns)

↑
column vectors correspond to exponents of M, L, T for non-repeating variables, BUT negative

Use a table

	REPEATING VARIABLES			NON-REPEATING VARIABLES (NEGATIVE EXPONENTS)	
	S	D	V	M	F ₀
M	1	0	0	-1	-1
L	-3	1	1	1	-1
T	0	0	-1	1	2

$\underbrace{\hspace{10em}}_{\underline{\underline{A}}}$
 $\underbrace{\hspace{10em}}_{\underline{\underline{B}}}$

Solution to $\underline{\underline{A}} \cdot \underline{\underline{X}} = \underline{\underline{B}} \Rightarrow \underline{\underline{X}} = \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}} =$ ~~$\begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$~~

~~$\underline{\underline{X}} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$~~

$$\underline{\underline{X}} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$$

$$\Pi_1 = S^{x_1} D^{y_1} V^{z_1} M = S^{-1} D^{-1} V^{-1} M = \frac{M}{\rho D V}$$

$$\Pi_2 = S^{x_2} D^{y_2} V^{z_2} F_0 = S^{-1} D^{-2} V^{-2} F_0 = \frac{F_0}{\rho D^2 V^2}$$

For WEIR CASE

	S	hw	g	W	H	B	Q	σ
M	1	0	0	0	0	0	0	-1
L	-3	1	1	-1	-1	-1	-3	0
T	0	0	-2	0	0	0	+1	+2

$\underbrace{\hspace{10em}}_{\underline{\underline{A}}}$
 $\underbrace{\hspace{10em}}_{\underline{\underline{B}}}$

$$\underline{\underline{X}} = \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -2.5 & -2 \\ 0 & 0 & 0 & -0.5 & -1 \end{bmatrix}$$

$$\Pi_1 = S^0 Hw^{-1} g^0 W = W/Hw$$

$$\Pi_2 = S^0 Hw^{-1} g^0 H = H/Hw$$

$$\Pi_3 = S^0 Hw^{-1} g^0 B = B/Hw$$

$$\Pi_4 = S^0 Hw^{-2.5} g^{-0.5} Q = \frac{Q}{\sqrt{g Hw^3}}$$

$$\Pi_5 = S^{-1} Hw^{-2} g^{-2} \sigma = \frac{\sigma}{\rho g Hw^2}$$