Ch.6. Momentum and Forces in Fluid Flow

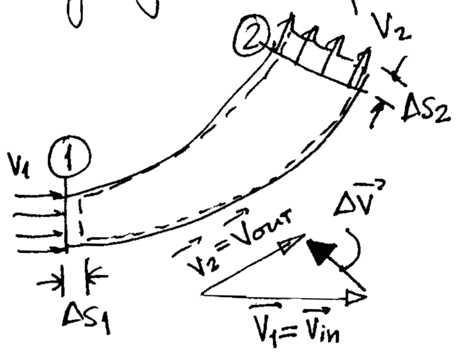
6.1. Development of the Momentum Principle

Start with a modified form of Newton's 2nd law: $\underbrace{\sum \vec{F} = \frac{d}{dt}(m\vec{V}_s)}_{\text{on a system}}$

$\boxed{\text{NOTE: If } m = \text{constant}, \ \frac{d}{dt}(m\vec{V}_s) = m\frac{d\vec{V}_s}{dt} = m\vec{a}.}$

Also, $\underbrace{\sum \vec{F} dt}_{\text{impulse}} = \underbrace{d(m\vec{V})_s}_{\Delta(\text{momentum})}$ ← impulse-momentum principle

Using Reynolds Transport Theorem (section 4.6)



$$\frac{dX_s}{dt} = \frac{dX_{cv}}{dt} + \frac{dX_{cv}^{out}}{dt} - \frac{dX_{cv}^{in}}{dt}$$ ← GENERAL UNSTEADY CASE

with $X = m\vec{V}$

For steady flow, $\frac{dX_{cv}}{dt} = 0$

$$\frac{dX_s}{dt} = \frac{dX_{cv}^{out}}{dt} - \frac{dX_{cv}^{in}}{dt}$$

For momentum principle: $\sum \vec{F} = \frac{d}{dt}(m\vec{V})\Big|_{cv}^{out} - \frac{d}{dt}(m\vec{V})\Big|_{cv}^{in}$

> For steady flow the net force on the fluid mass is equal to the net rate of OUTFLOW of momentum across the control surface.

- Select a C.V. such that $V \perp$ C.S. where it cuts the flow
- V = constant where C.S. cuts flow $\Rightarrow \frac{d}{dt}(m\vec{V})_1 = \frac{dm_1}{dt}\vec{V}_1 = \dot{m}_1\vec{V}_1 = \rho_1 Q_1 \vec{V}_1$

Thus, for steady flow $\sum \vec{F} = \dot{m}_2\vec{V}_2 - \dot{m}_1\vec{V} = \rho_2 Q_2 \vec{V}_2 - \rho_1 Q_1 \vec{V}_1$

Since the flow is steady $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho_1 Q_1 = \rho_2 Q_2 = \rho Q \Rightarrow$

$\rho_2 Q_2 \vec{V}_2 - \rho_1 Q_1 \vec{V}_1 = \rho Q(\vec{V}_2 - \vec{V}_1) = \rho Q \Delta\vec{V}$, with $\Delta\vec{V} = \vec{V}_2 - \vec{V}_1 = \vec{V}_{out} - \vec{V}_{in}$

$\Rightarrow$ MOMENTUM (or impulse-momentum) principle

$$\boxed{\sum \vec{F} = \dot{m}\Delta\vec{V} = \rho Q \Delta\vec{V} = \rho Q(\vec{V}_2 - \vec{V}_1)}$$

$\sum \vec{F}$ = vector sum of all forces acting on fluid mass

- Gravity forces
- Shear forces
- Pressure forces
  - From fluid mass
  - From solid boundaries

$\rho Q \Delta \vec{V}$ = change in momentum per unit time

$$\boxed{\sum \vec{F} = \dot{m}\,\Delta \vec{V} = \rho Q \Delta \vec{V} = \rho Q(\vec{V}_2 - \vec{V}_1)}$$

Using scalar components:

$$\boxed{\begin{aligned} \sum F_x &= \dot{m}\,\Delta V_x = \rho Q \Delta V_x = \rho Q (V_{2x} - V_{1x}) \\ \sum F_y &= \dot{m}\,\Delta V_y = \rho Q \Delta V_y = \rho Q (V_{2y} - V_{1y}) \\ \sum F_z &= \dot{m}\,\Delta V_z = \rho Q \Delta V_z = \rho Q (V_{2z} - V_{1z}) \end{aligned}}$$

<u>NOTE</u>: read section 6.2 → Navier-Stokes equations: momentum equations for a fluid particle in three coordinate directions (real fluids), e.g., (6.8c)

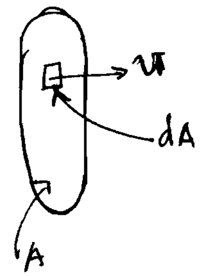
$$\underbrace{\underbrace{-\rho g}_{\frac{\text{gravitational forces (weight)}}{\text{volume}}} \underbrace{- \frac{\partial p}{\partial z}}_{\frac{\text{pressure forces}}{\text{volume}}} + \underbrace{\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}_{\frac{\text{viscous forces}}{\text{volume}}}} = \underbrace{\rho}_{\frac{\text{mass}}{\text{volume}}\ (m/\forall)} \underbrace{\left[\underbrace{\frac{\partial w}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{convective acceleration}}\right]}_{\text{acceleration } (a_z)}$$

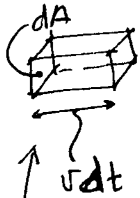
$$\frac{\sum F_z}{\text{volume}} = \frac{m}{\text{volume}} \cdot a_z \Rightarrow \sum F_z = m a_z$$

i.e., Newton's 2nd law

- lengthy and involved derivation
- 2nd-order nonlinear partial differential eqn (PDE)
- No general solution available
- numerical solution ⟹ computational fluid dynamics (CFD)

63. Momentum correction Factor





$$\left\{\begin{matrix}\text{momentum flux}\\ \text{across } dA\end{matrix}\right\} = d\dot{m}\cdot u = (\rho u\,dA)\,u = \rho u^2 dA$$

volume
$d\forall = u\,dt\,dA$

mass
$dm = \rho\, d\forall = \rho u\, dt\, dA$

$\frac{dm}{dt} = d\dot{m} = \rho u\, dA$

$$\left\{\begin{matrix}\text{ACTUAL}\\ \text{momentum}\\ \text{flux across}\\ A\end{matrix}\right\} = \int_A \rho u^2 dA$$

$$\left\{\begin{matrix}\text{momentum}\\ \text{flux with}\\ \text{average velocity}\end{matrix}\right\} = \int_A \rho V^2 dA = \rho V^2 A$$

$$\left\{\begin{matrix}\text{actual momentum}\\ \text{flux across } A\end{matrix}\right\} = \beta\cdot\left\{\begin{matrix}\text{momentum flux}\\ \text{with average velocity}\end{matrix}\right\}$$

$$\int_A \rho u^2 dA = \beta \rho V^2 A$$

with constant ρ: $\boxed{\beta = \frac{1}{AV^2}\int_A u^2 dA}$

For laminar flow in circular pipe, $\beta = \frac{4}{3} = 1.33$

For turbulent flow in circular pipe, $\beta = 1.005$ to 1.05, i.e., $\beta \simeq 1.0$

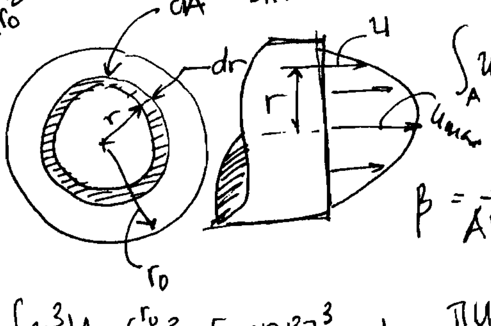
E.g. for laminar flow $Q = \int_A u\,dA = \int_0^{r_0} U_{max}\left(1-\left(\frac{r}{r_0}\right)^2\right)\cdot 2\pi r\,dr$

$u = U_{max}\left(1-\left(\frac{r}{r_0}\right)^2\right)$

$Q = \frac{\pi}{2} U_m r_0^2$, $V = \frac{Q}{A} = \frac{\pi U_m r_0^2}{2\pi r_0^2} = \frac{U_{max}}{2}$

$A = \pi r_0^2$

$dA = 2\pi r\,dr$



$$\int_A u^2 dA = \int_0^{r_0} U_{max}^2\left(1-\left(\frac{r}{r_0}\right)^2\right)^2 2\pi r\,dr = \frac{U_{max}^2 r_0^2 \pi}{3}$$

$$\beta = \frac{1}{AV^2}\int_A u^2 dA = \frac{1}{\cancel{\pi r_0^2}\frac{\cancel{U_m^2}}{4}}\cdot\frac{\cancel{U_m^2}\cancel{r_0^2}\cancel{\pi}}{3} = \frac{4}{3}$$

momentum correction factor

also, $\int_A u^3 dA = \int_0^{r_0} U_{max}^3\left[1-\left(\frac{r}{r_0}\right)^2\right]^3 2\pi r\,dr = \frac{\pi U_{max}^3 r_0^2}{4}$, $\alpha = \frac{1}{AV^3}\int_A u^3 dA = 2.0$

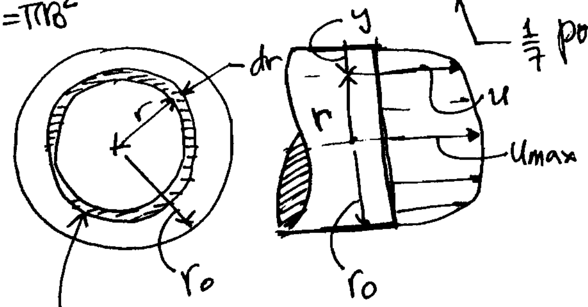
kinetic energy correction factor

One turbulent flow possibility in pipes (see eq. 8.49)

Smooth pipe, $\frac{u}{U_{max}} = \left(\frac{y}{r_0}\right)^{1/7}$, y = measured from wall

$\frac{1}{7}$ power law

$A = \pi r_0^2$



note: $r + y = r_0$, thus

$$u = U_{max}\left(\frac{r_0 - r}{r_0}\right)^{1/7} \text{ or}$$

$$u = U_{max}\left(1 - \frac{r}{r_0}\right)^{1/7}$$

$dA = 2\pi r\,dr$

$$Q = \int_A u\,dA = \int_0^{r_0} U_{max}\left(1 - \frac{r}{r_0}\right)^{1/7} 2\pi r\,dr = \frac{49}{60}\pi U_m r_0^2, \quad V = \frac{Q}{A} = \frac{\frac{49}{60}\pi U_m r_0^2}{\pi r_0^2}$$

$$V = \frac{49}{60} U_m$$

$$\int u^2 dA = \int_0^{r_0} U_{max}^2\left(1 - \frac{r}{r_0}\right)^{2/7} 2\pi r\,dr = \frac{49}{72}\pi U_m^2 r_0^2$$

$$\beta = \frac{1}{AV^2}\int_A u^2 dA = \frac{1}{\pi r_0^2\left(\frac{49}{60}U_m\right)^2}\left(\frac{49}{72}\pi U_m^2 r_0^2\right) = \frac{50}{49} = 1.020$$

momentum correction factor

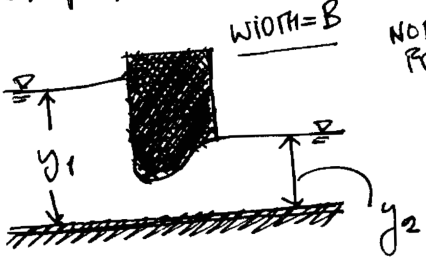
$$\int u^3 dA = \int_0^{r_0} U_{max}^3\left(1 - \frac{r}{r_0}\right)^{3/7} 2\pi r\,dr = \frac{49}{85}\pi U_m^3 r_0^2$$

$$\alpha = \frac{1}{AV^3}\int_A u^3 dA = \frac{1}{\pi r_0^2\left(\frac{49}{60}U_m\right)^3} \cdot \frac{49}{85}\pi U_m^3 r_0^2 = \frac{43200}{40817} = 1.058$$

Kinetic energy correction factor

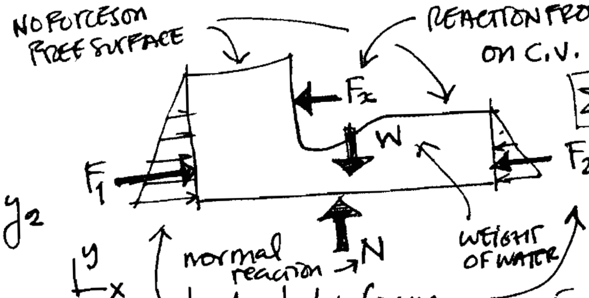
6.4. Applications of the Momentum Principle → Open-Channel flow cases

See Sample Problem 6.1



FORCES ON THE C.V.

$$\boxed{\Sigma F_x = F_1 - F_x - F_2}$$

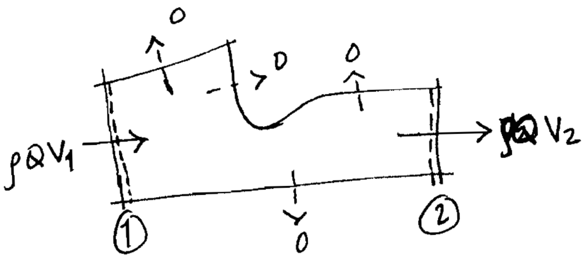


$$\boxed{\Sigma F_y = N - W}$$

$F_1 = \frac{1}{2}\gamma B y_1^2$

$F_2 = \frac{1}{2}\gamma B y_2^2$

MOMENTUM FLUXES



$$\boxed{\Delta(\rho Q V)_x = \rho Q V_2 - \rho Q V_1}$$

$$\boxed{\Delta(\rho Q V)_y = 0}$$ [NO FLOW IN VERTICAL DIRECTION]

Momentum Principle components

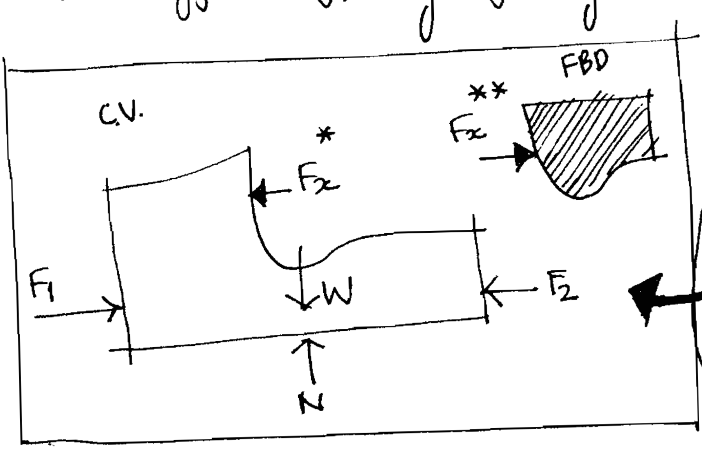
$$\Sigma F_x = \Delta(\rho Q V)_x \Rightarrow \tfrac{1}{2}\gamma B(y_1^2 - y_2^2) - F_x = \rho Q (V_2 - V_1)$$

$$F_x = \underbrace{\tfrac{1}{2}\gamma B(y_1^2 - y_2^2)}_{\text{KNOWN}} - \underbrace{\rho Q (V_2 - V_1)}_{\text{UNKNOWN}}$$

Use continuity: $Q = \cancel{B} y_1 V_1 = \cancel{B} y_2 V_2 \Rightarrow y_1 V_1 = y_2 V_2$

and energy: $y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g}$

} solve for V_1, V_2, calculate Q, then



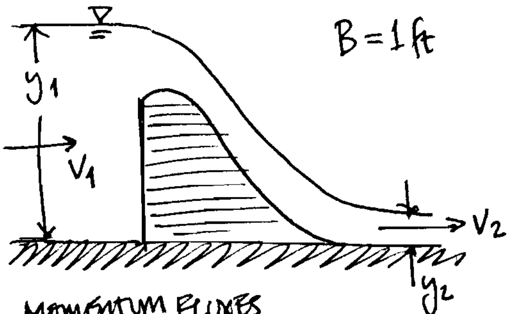
$$\boxed{\begin{gathered}\Sigma F_y = \Delta(\rho Q V)_y \\ W - N = 0 \Rightarrow N = W\end{gathered}}$$

* ← F_x : Force applied by obstacle on C.V.

** F_x → : Force applied by flow on obstacle

EXERCISES.

6.44. Flow occurs over a spillway of constant section as shown in Fig. X6.44. Assuming ideal flow, determine the resultant horizontal force on the spillway per foot of spillway width (perpendicular to the spillway section), given that $y_1 = 4.2$ ft and $y_2 = 0.7$ ft.

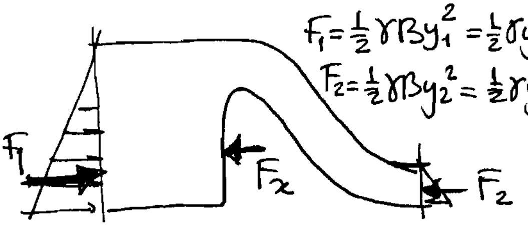


$B = 1$ ft

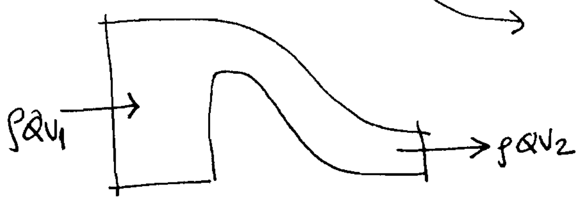
FORCES $\quad \sum F_x = F_1 - F_x - F_2$

$F_1 = \frac{1}{2}\gamma B y_1^2 = \frac{1}{2}\gamma y_1^2$

$F_2 = \frac{1}{2}\gamma B y_2^2 = \frac{1}{2}\gamma y_2^2$



MOMENTUM FLUXES



$B=1$

$$\Delta(\rho Q V)_x = \rho Q V_2 - \rho Q V_1 = \rho Q (V_2 - V_1)$$

impulse-momentum: $\sum F_x = \Delta(\rho Q V)_x$

$$\frac{1}{2}\gamma y_1^2 - F_x - \frac{1}{2}\gamma y_2^2 = \rho Q (V_2 - V_1)$$

$$F_x = \frac{\gamma}{2}(y_1^2 - y_2^2) - \rho Q (V_2 - V_1)$$

$$\frac{\gamma}{g}$$

Continuity: $Q = \cancel{B} y_1 V_1 = \cancel{B} y_2 V_2$

$$y_1 V_1 = y_2 V_2 \Rightarrow \frac{V_2}{V_1} = \frac{y_1}{y_2}$$

Energy: $y_1 + \frac{V_1^2}{2g} = y_2 + \frac{V_2^2}{2g} \Rightarrow \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = y_2 - y_1 \Rightarrow \frac{V_1^2}{2g}\left[1 - \left(\frac{V_2}{V_1}\right)^2\right] = y_2 - y_1$

$$\Rightarrow \frac{V_1^2}{2g}\left[1 - \left(\frac{y_1}{y_2}\right)^2\right] = y_2 - y_1 \Rightarrow V_1 = \sqrt{\frac{2g(y_2 - y_1)}{1 - (y_1/y_2)^2}}$$

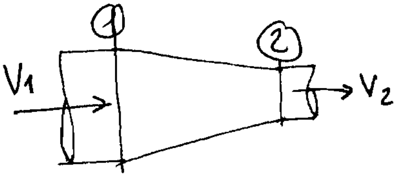
$$V_1 = \sqrt{\frac{2 \times 32.2\ \text{ft/s}^2 \times (0.7 - 4.2)\ \text{ft}}{1 - (4.2/0.7)^2}} = 2.54\ \text{ft/s}, \quad V_2 = \frac{y_1}{y_2} V_1 = \frac{4.2}{0.7} \times 2.54 = 15.22\ \frac{\text{ft}}{\text{s}}$$

$$Q = B y_1 V_1 = (1\text{ft})(4.2\text{ft})\left(2.54\ \frac{\text{ft}}{\text{s}}\right) = 10.668\ \text{ft}^3/\text{s}$$

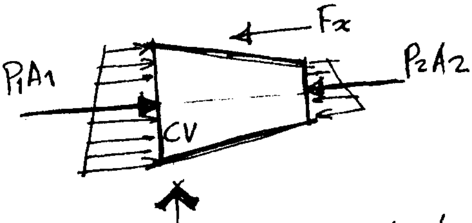
$$F_x = \frac{62.4\ \text{lb/ft}^3}{2} \times (4.2^2 - 0.7^2)\ \text{ft}^2 \times 1\text{ft} - \frac{62.4\ \text{lb/ft}^3}{32.2\ \text{ft/s}^2} \times (15.22 - 2.54)\frac{\text{ft}}{\text{s}} \times 10.668\ \frac{\text{ft}^3}{\text{s}}$$

$$\boxed{F_x = 272.94\ \text{lb}}$$

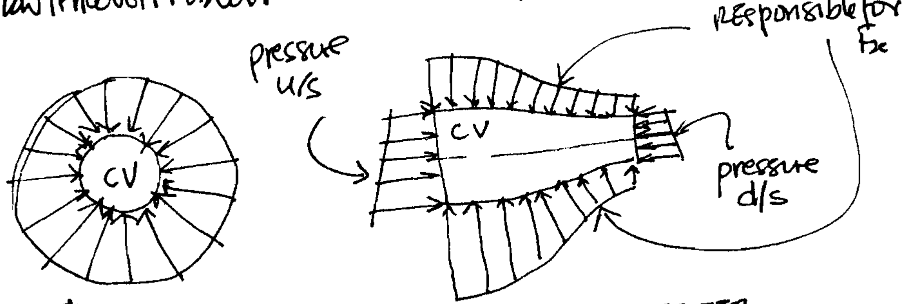
6.5. Force on Pressure Conduit [e.g. reducer]



FLOW THROUGH REDUCER



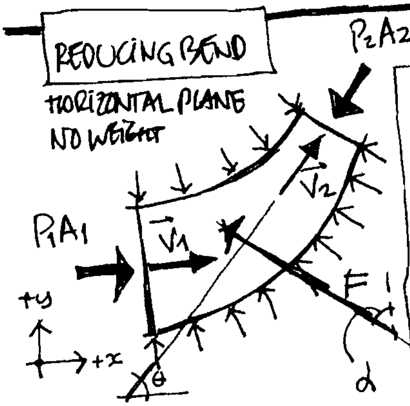
FORCES ON CONTROL VOLUME

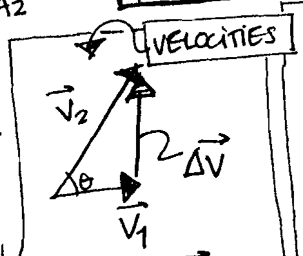


PRESSURE DISTRIBUTION ON THE FLUID IN A REDUCER

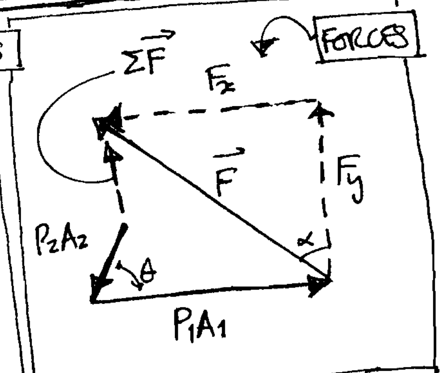
$$\Sigma F_x = \Delta(\rho Q V)_x \Rightarrow P_1A_1 - F_x - P_2A_2 = \rho Q (V_2 - V_1)$$

$$\boxed{F_x = P_1A_1 - P_2A_2 - \rho Q (V_2 - V_1)}$$





$$\vec{V_2} = \vec{V_1} + \Delta\vec{V}$$
$$\Delta\vec{V} = \vec{V_2} - \vec{V_1}$$



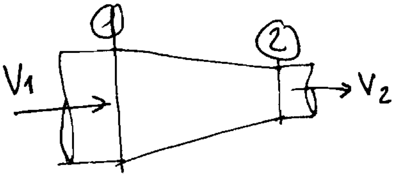
$\boxed{x\text{-component:}}$ $\Sigma F_x = \Delta(\rho Q V)_x \Rightarrow P_1A_1 - F_x - P_2A_2\cos\theta = \rho Q (V_{2x} - V_{1x})$ | $V_{1x} = V_1$, $V_{2x} = V_2\cos\theta$

$$\boxed{F_x = P_1A_1 - P_2A_2\cos\theta - \rho Q (V_2\cos\theta - V_1)}$$

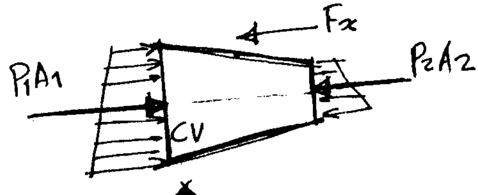
$\boxed{y\text{-component:}}$ $\Sigma F_y = \Delta(\rho Q V)_y \Rightarrow -P_2A_2\sin\theta + F_y = \rho Q (V_{2y} - V_{1y})$ | $V_{1y} = 0$, $V_{2y} = V_2\sin\theta$

$$\boxed{F_y = P_2A_2\sin\theta + \rho Q V_2\sin\theta}$$

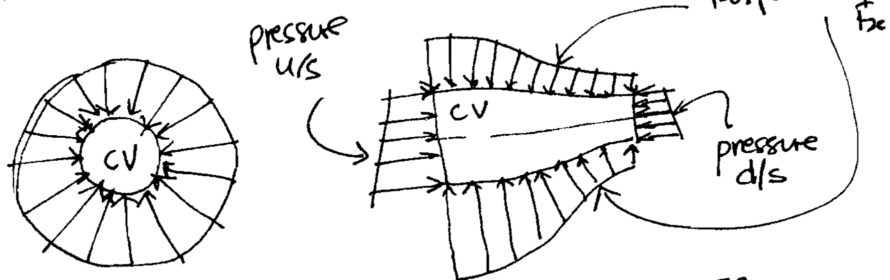
6.5. Force on Pressure Conduit [e.g. reducer]



FLOW THROUGH REDUCER



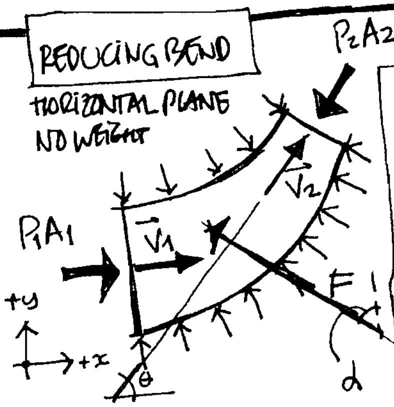
FORCES ON CONTROL VOLUME

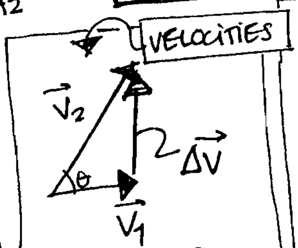


PRESSURE DISTRIBUTION ON THE FLUID IN A REDUCER

$$\sum F_x = \Delta(\rho Q V)_x \Rightarrow P_1A_1 - F_x - P_2A_2 = \rho Q(V_2 - V_1)$$

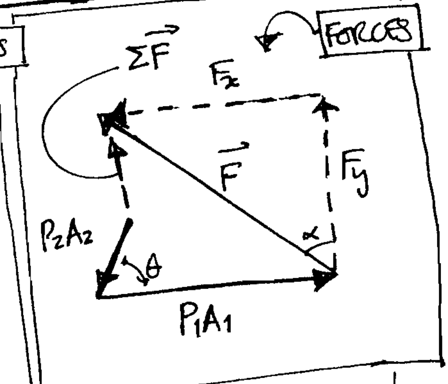
$$\boxed{F_x = P_1A_1 - P_2A_2 - \rho Q(V_2 - V_1)}$$





$\vec{V_2} = \vec{V_1} + \vec{\Delta V}$

$\vec{\Delta V} = \vec{V_2} - \vec{V_1}$



x-component: $\sum F_x = \Delta(\rho Q V)_x \Rightarrow P_1A_1 - F_x - P_2A_2\cos\theta = \rho Q(V_{2x} - V_{1x})$ | $V_{1x} = V_1$, $V_{2x} = V_2\cos\theta$

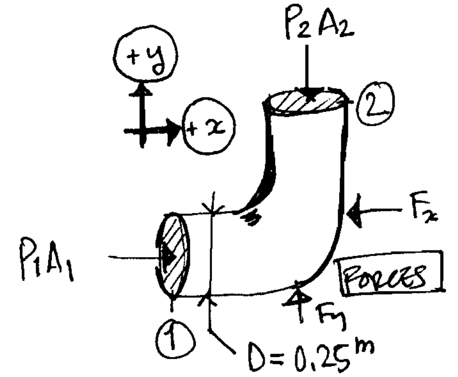
$$\boxed{F_x = P_1A_1 - P_2A_2\cos\theta - \rho Q(V_2\cos\theta - V_1)}$$

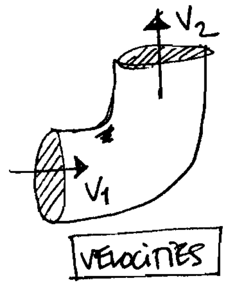
y-component: $\sum F_y = \Delta(\rho Q V)_y \Rightarrow -P_2A_2\sin\theta + F_y = \rho Q(V_{2y} - V_{1y})$ | $V_{1y} = 0$, $V_{2y} = V_2\sin\theta$

$$\boxed{F_y = P_2A_2\sin\theta + \rho Q V_2\sin\theta}$$

EXERCISES

6.5.5. Water under a gage pressure of 350 kPa flows with a velocity of 5 m/s through a right-angled bend that has a uniform diameter of 250 mm. The bend lies in a horizontal plane, and water enters from the west and leaves toward the north. Assuming no drop in pressure, what is the magnitude and direction of the resultant force acting on the bend?





$P_1 = P_2 = 350\ \text{kPa}$

$V_1 = V_2 = 5\ \text{m/s}$

$A_1 = A_2 = \frac{\pi D^2}{4} = \frac{\pi \times 0.25^2}{4}$

$= 4.91 \times 10^{-2}\ \text{m}^2$

$$\xrightarrow{+} \Sigma F_x = \Delta(\rho Q V)_x \Rightarrow -F_x + P_1 A_1 = \rho Q (V_{2x} - V_{1x}) = \rho Q (0 - V_1)$$

$$F_x = P_1 A_1 + \rho Q V_1 = \left(350 \times 10^3 \frac{N}{m^2}\right)(4.91 \times 10^{-2}\ m^2) + \left(1000 \frac{kg}{m^3}\right)\left(0.2454 \frac{m^3}{s}\right)(5\ m/s)$$

$$Q = V \frac{\pi D^2}{4} = \left(5 \frac{m}{s}\right) \times \frac{\pi \times (0.25 m)^2}{4}$$

$$Q = 0.2454\ m^3/s$$

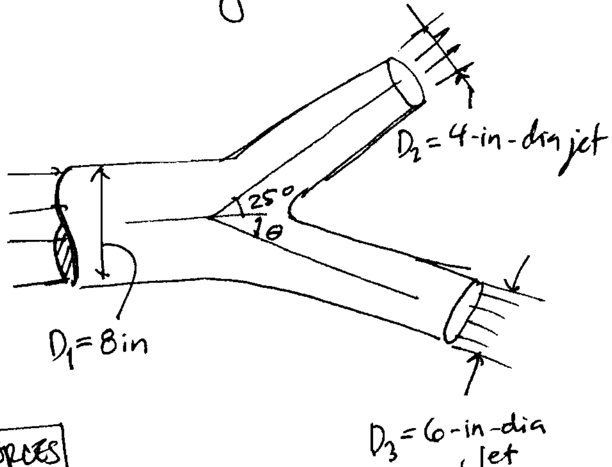
$$\boxed{F_x = 18412\ N = 18.41\ kN}$$

$$\overset{+}{\uparrow} \Sigma F_y = \Delta(\rho Q V)_y \Rightarrow F_y - P_2 A_2 = \rho Q (V_{2y} - V_{1y}) = \rho Q (V_2 - 0)$$

$$F_y = P_2 A_2 + \rho Q V_2 = \left(350 \times 10^3 \frac{N}{m^2}\right)(4.91 \times 10^{-2}\ m^2) + \left(1000 \frac{kg}{m^3}\right)\left(0.2454 \frac{m^3}{s}\right)(5\ m/s)$$

$$\boxed{F_y = 18412\ N = 18.41\ kN}$$

6.5.3. In Fig. X6.5.3, both nozzles discharge water horizontally into the atmosphere at 30 fps. Find θ so that the resultant force on the unit is along the axis of the 8-in-diameter pipe.



$$A_1 = \frac{\pi(8/12)^2}{4} = 0.349\ ft^2$$

$$A_2 = \frac{\pi(4/12)^2}{4} = 0.087\ ft^2$$

$$A_3 = \frac{\pi\times(6/12)^2}{4} = 0.196\ ft^2$$

continuity

$$Q_1 = Q_2 + Q_3$$

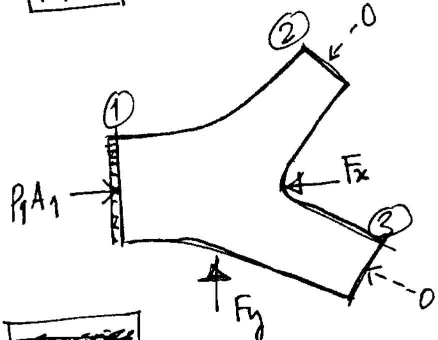
$$V_1\cdot\frac{\cancel{\pi} D_1^2}{\cancel{4}} = V_2\cdot\frac{\cancel{\pi} D_2^2}{\cancel{4}} + V_3\frac{\cancel{\pi}\cdot D_3^2}{\cancel{4}}$$

$$V_1 D_1^2 = V_2 D_2^2 + V_3 D_3^2$$

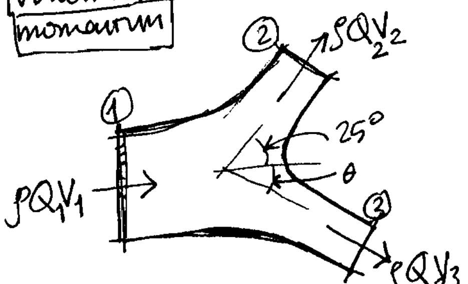
$$64V_1 = 16V_2 + 36V_3$$

$$\boxed{16V_1 = 4V_2 + 9V_3}$$

FORCES



~~VELOCITIES~~ momentum



momentum in x:

$$P_1A_1 - F_x = \rho(Q_2V_2\cos 25° + \cancel{\rho} Q_3V_3\cos\theta - Q_1V_1)$$

$$QV = (V\cdot A)\cdot V = V^2 A$$

momentum in y:

$$F_y = \rho(Q_2V_2\sin 25° - \rho Q_3V_3\sin\theta) = 0$$

$$F_y = \boxed{\cancel{\rho} A_2V_2^2\sin 25° - \cancel{\rho} A_3V_3^2\sin\theta = 0}$$

Energy $z_1 = z_2 = z_3$

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} = \frac{V_2^2}{2g}$$

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} = \frac{V_3^2}{2g}$$

$V_2 = V_3$

$$\sin\theta = \frac{A_2}{A_3}\left(\frac{V_2}{V_3}\right)^2\sin 25°$$

$$\sin\theta = \frac{0.087}{0.196}\sin 25° = 0.1876$$

$$\boxed{\theta = 10.81°}$$