Notes on Liquid Masses Subjected to Acceleration

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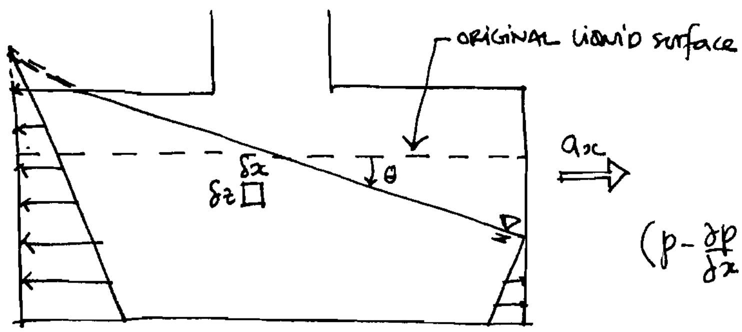
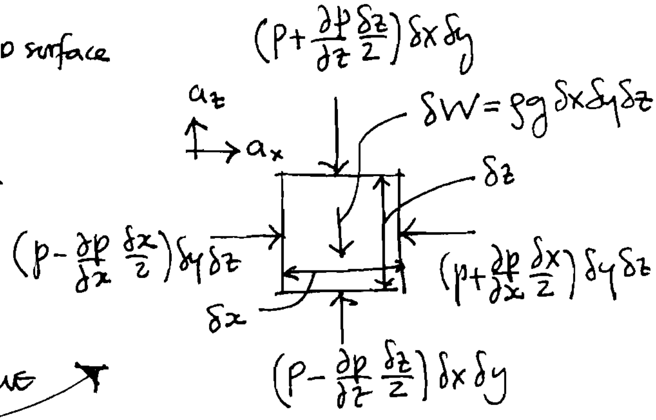
for
CEE 3500 – CEE Fluid Mechanics

Liquid Masses subjected to acceleration

- Body of liquid transported in a tank at constant speed $\Rightarrow$ conditions similar to liquid at rest
- If the tank accelerates in the horizontal direction:
  1. No relative motion between particles
  2. Free surface (if any) tilts with respect to the horizontal





FREE-BODY DIAGRAM OF ELEMENTAL FLUID PARTICLE UNDER ACCELERATION

$$\overset{+}{\rightarrow}\sum F_x = ma_x \Rightarrow \left(p-\frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z-\left(p+\frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z=\rho\,\delta x\,\delta y\,\delta z\,a_x \Rightarrow \boxed{\frac{\partial p}{\partial x}=-\rho a_x}$$

$$\overset{+}{\uparrow}\sum F_z = ma_z \Rightarrow \left(p-\frac{\partial p}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y-\rho g\,\delta x\,\delta y\,\delta z-\left(p+\frac{\partial p}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y=\rho\,\delta x\,\delta y\,\delta z\,a_x$$

$$\Rightarrow \boxed{\frac{\partial p}{\partial z}=-\rho(a_z+g)}$$

NOTE: if no vertical acceleration present, $a_z=0$, conditions are hydrostatic, $\frac{\partial p}{\partial z}=-\rho g$

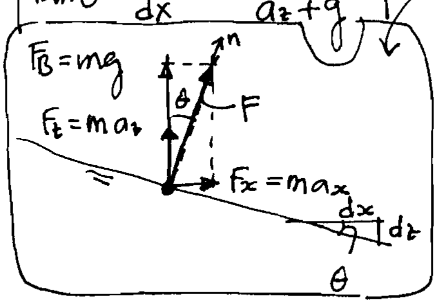
Planes of constant pressure: For $p=p(x,z)$, $dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial z}dz$, in general.
If $p(x,y)=$ constant, then $dp=0$ or $\frac{\partial p}{\partial z}dz=-\frac{\partial p}{\partial x}dx \Rightarrow -\cancel{\rho}(a_z+g)dz=\cancel{\rho}a_x dx$

$$\boxed{\frac{dz}{dx}=-\frac{a_x}{a_z+g}}$$

Since the free surface is a plane of constant pressure, the slope of the free-surface is

$$\boxed{\tan\theta=\frac{dz}{dx}=-\frac{a_x}{a_z+g}}$$

Forces on a particle in the surface. Resultant is $\perp$ to surface



and

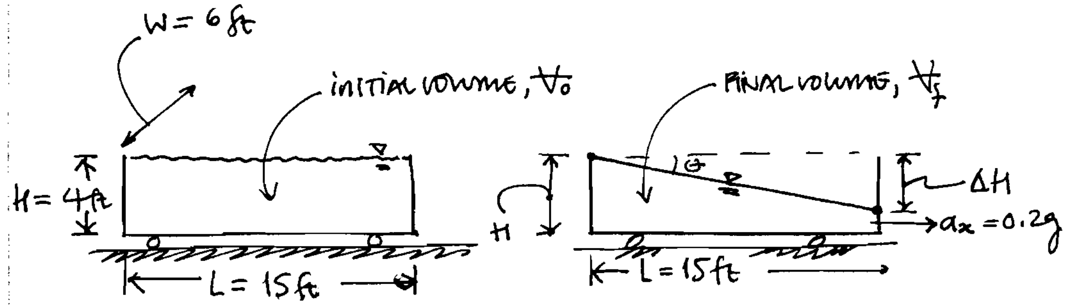
$$\boxed{\frac{\partial p}{\partial n}=-\rho\sqrt{a_x^2+(a_z+g)^2}}$$

If $a_x=a_z=0 \Rightarrow n=z$, and we recover hydrostatic conditions

$$\frac{\partial p}{\partial z}=-\rho g$$

EXAMPLES

3.10.5. Suppose that the tank in Fig. 3.24 is rectangular and completely open at the top. It is 15 ft long, 6 ft wide, and 4 ft deep. If it is initially filled to the top, how much liquid will be spilled if it is given a horizontal acceleration $a_x = 0.2g$ in the direction of its length?



$H' = H - \Delta H$
$= 4 - 2.94$
$= 1.06$ ft

The slope of the free surface will be $\tan\theta = -\dfrac{a_x}{a_z + g} = -\dfrac{0.2g}{0+g} = -0.2$

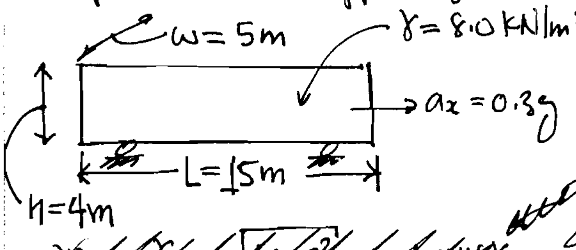
$\Rightarrow \theta = -11.31°$, the drop in vertical distance for the L = 15 ft length of the tank is, $\Delta H = L\cos\theta = 15\text{ ft} \times \sin(-11.31°) = -2.94$ ft

volumes: $\forall_o = L \cdot H \cdot W = (15\text{ ft}) \times (4\text{ ft}) \times (6\text{ ft}) = 360\text{ ft}^3$

$\forall_f = \frac{1}{2}(H + H') \cdot L \cdot W = \frac{1}{2}(4.00 + 1.06)\text{ft} \times (15\text{ ft}) \times (6\text{ ft}) = 227.7\text{ ft}^3$

spilled volume: $\underline{\Delta\forall = \forall_o - \forall_f = 360\text{ ft}^3 - 227.7\text{ ft}^3 = 132.3\text{ ft}^3}$

3.10.8 If the tank of Exerc. 3.10.6 is closed at the top and is completely filled, what must be the pressure difference between the left-hand end at the top and the right-hand end at the top if the liquid has a specific weight of 8.0 kN/m³ and the horizontal acceleration is $a_x = 0.3g$? Sketch planes of equal pressure, indicating their magnitude; assume zero pressure in the upper right-hand corner.



$\tan\theta = -\dfrac{a_x}{a_z + g} = -\dfrac{0.3g}{0+g} = -0.3, \ \theta = -16.70°$

$\cancel{\dfrac{dz}{dx} = -0.3}$

$\cancel{\dfrac{dp}{dn} = -\dfrac{\gamma}{g}\sqrt{a_x^2 + (a_z + g)^2} = -\dfrac{\gamma}{g}\sqrt{(0.3g)^2 + (0+g)^2}}$

$\cancel{\dfrac{dp}{dn} = -\dfrac{\gamma}{g} \cdot g \cdot \sqrt{1 + 0.3^2} = -1.044\gamma}$

$dz = -0.3dx \Rightarrow \boxed{z = -0.3x + C}$ ← GENERAL EQ. of CONSTANT PRESSURE planes

~~for x = 15 and z = 4, p = 0~~ $\Rightarrow$ ~~C = z + 0.3x = 4 + 0.3 × 15 = 8.5~~

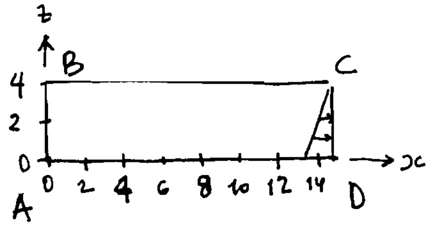
~~x = 15 and z = 2, p = γ·2 = 16 kN/m² = 16 kPa~~

~~C = z + 0.3x = 2 + 0.3 × 15 = 6.5~~

~~x = 15 and z = 0, p = γ·4 = 32 kPa~~

~~C = z + 0.3x = 0 + 0.3 × 15 = 4.5~~

using pressures on CD, knowing that $p_C = 0$



Integrating $\frac{\partial p}{\partial x} = -\rho a_x = -\frac{\gamma}{g} \times 0.3g = -0.3\gamma$ with respect to x:

$$p(x,z) = -0.3\gamma x + K(z) \Rightarrow \frac{\partial p}{\partial z} = \frac{dK}{dz}, \text{ also } \frac{\partial p}{\partial z} = -\rho g = -\gamma$$

thus, $\frac{dK}{dz} = -\gamma \Rightarrow K(z) = -\gamma z + M$, and

$$p(x,z) = -0.3\gamma x - \gamma z + M$$

With $p(x,z) = 0$ at $x = 15^m$, $z = 4^m \Rightarrow 0 = -0.3 \cdot \gamma \cdot 15 - \gamma \cdot 4 + M$

$\Rightarrow M = 8.5\gamma$, thus $\boxed{p(x,z) = \gamma(8.5 - 0.3x - z)}$ ← pressure distribution

max. value at $x=0, z=0 \Rightarrow p = 8.5\gamma$

or, $\boxed{\frac{p}{\gamma} = 8.5 - 0.3x - z}$

$0 \le \frac{p}{\gamma} \le 8.5$

For constant p (or p/γ), the planes of constant pressure are

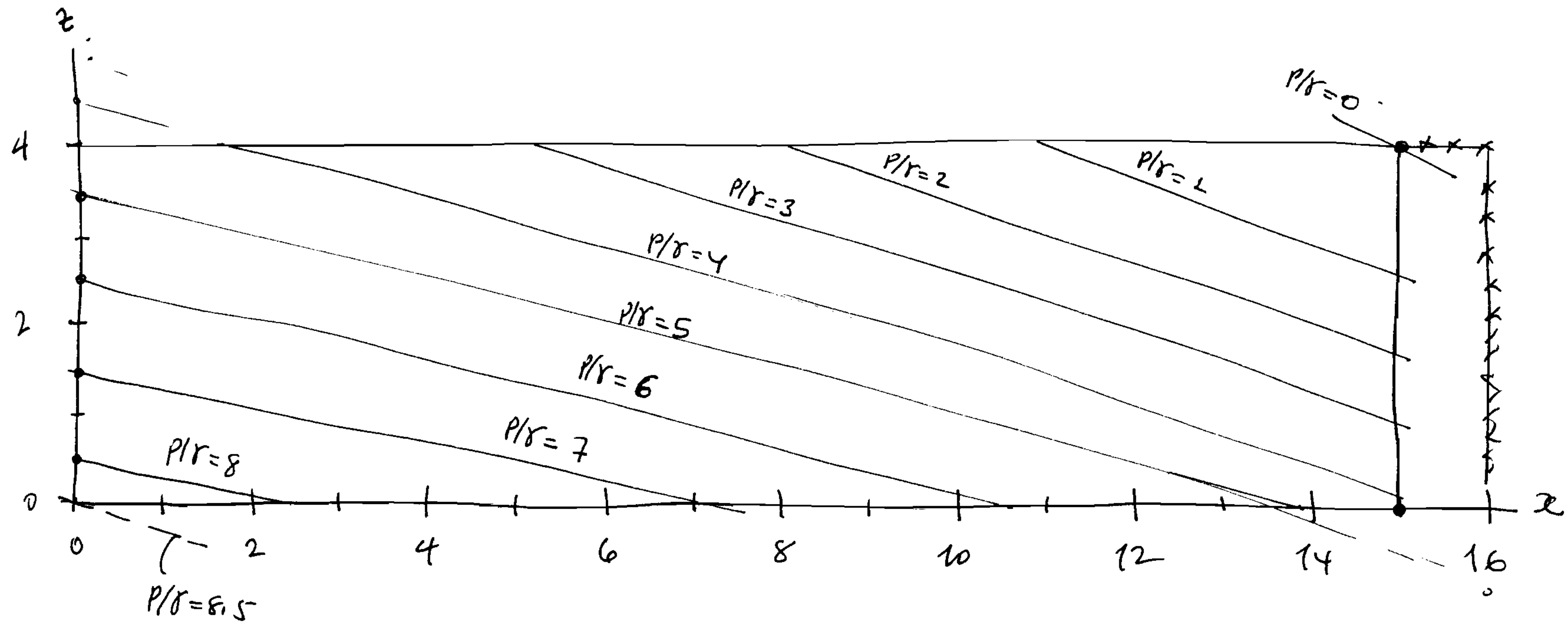
$$\boxed{z = -0.3x + \left(8.5 + \frac{-p}{\gamma}\right)} \rightarrow 0.3x = -z + \left(8.5 - \frac{p}{\gamma}\right)$$

$$x = \frac{1}{0.3}\left(-z + \left(8.5 - \frac{p}{\gamma}\right)\right)$$

(SEE GRAPH in next page)

p/γ (m)	$z(x)$	VALUES OF z for $x=0$	$x=15$
0.0	$-0.3x + 8.5$	8.5	4.0
1.0	$-0.3x + 7.5$	7.5	3.0
2.0	$-0.3x + 6.5$	6.5	2.0
3.0	$-0.3x + 5.5$	5.5	1.0
4.0	$-0.3x + 4.5$	4.5	0.0
5.0	$-0.3x + 3.5$	3.5	−1.0
6.0	$-0.3x + 2.5$	2.5	−2.0
7.0	$-0.3x + 1.5$	1.5	−3.0
8.0	$-0.3x + 0.5$	0.5	−4.0
8.5	$-0.3x$	0.0	−4.5

p/γ (m)	$x(z) = \ldots$	$z=0$	$z=4$
0.0	$-3.33z + 28.33$	28.33	15.0
1.0	$-3.33z + 25$	25.0	11.66
2.0	$-3.33z + 21.66$	21.66	8.33
3.0	$-3.33z + 18.33$	18.33	5.00
4.0	$-3.33z + 15.00$	15.00	1.66
5.0	$-3.33z + 11.66$	11.66	−1.66
6.0	$-3.33z + 8.33$	8.33	−5.00
7.0	$-3.33z + 5.00$	5.00	−8.33
8.0	$-3.33z + 1.66$	1.66	−11.66
8.5	$-3.33z$	0.00	−13.33


z
x
0
2
4
6
8
10
12
14
16
P/γ=0
P/γ=1
P/γ=2
P/γ=3
P/γ=4
P/γ=5
P/γ=6
P/γ=7
P/γ=8
P/γ=8,5